Partners Universal International Innovation Journal (PUIIJ)
Volume: 04 Issue: 03 | May-June 2026 | ISSN: 2583-9675 | www.puiij.com

Cubic Expectation of Cubic Mean of Random Variables

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Abstract - Concepts of cubic expectation of random variable was introduced with formulating its definition and deriving some of its elementary properties in an earlier study. One property of cubic expectation which relates the cubic expectation and cubic mean of random variables has been derived in this study. Derivation of the property has been presented in this article.

Keywords: Random Variable, Cubic Expectation, Cubic Mean, Property.

1. INTRODUCTION

As per the concept in the literature of statistics, expectation in general is defined as the theoretical average of the possible values assumed by a variable more specifically as the weighted average of its all possible values with their respective probabilities as the corresponding weights [1 , 16 , 17 , 20 , 25]. At the beginning of its development, it had been defined on the basis of weighted arithmetic mean terming it as mathematical expectation [1 , 16 , 17 , 20 , 25] which later was termed as arithmetic expectation [4 , 8]. In some studies done later, concept of expectation had been introduced and consequently defined using the concepts of geometric mean [1 , 3], harmonic mean [1 , 3], quadratic mean [18 , 19] and cubic mean [13 , 22] which were termed as geometric expectation [4 , 8], harmonic expectation [4 , 8], quadratic expectation [7 , 8] and cubic expectation [10] respectively.

Mathematical expectation is closely related to the concept of unbiasedness of estimators in the theory of statistical estimation [14 , 15 , 16 , 23]. Due to its importance, several studies had also been done of identifying the properties of the above five concepts of expectations where it was possible to identify some of their properties [2 , 5 – 9 , 11 , 12 , 17 , 20 , 21 , 24 , 26]. One property of cubic expectation which relates the cubic expectation and cubic mean of random variables has been derived in this study. Derivation of the property has been presented in this article.

2. CUBIC EXPECTATION

Let us the notation X to denote a real valued random variable.

Cubic Expectation of Random Variable

If X is finite discrete assuming the values

$$x_1 , x_2 , \ldots\ldots , x_N$$

which correspond to the probabilities

$$p_1 , p_2 , \ldots\ldots , p_N ,$$

then the cubic expectation of X, denoted by $E_C(X)$, is defined by

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$$E_c(X) = \left(\sum_{i=1}^{N} p_i x_{i3}\right)^{\frac{1}{3}}$$

and if X is denumerable discrete assuming the values

$$x_1, x_2, \ldots\ldots\ldots$$

which correspond to the probabilities

$$p_1, p_2, \ldots\ldots\ldots,$$

then,

$$E_c(X) = \left(\sum_{i=1}^{\infty} p_i x_{i3}\right)^{\frac{1}{3}}$$

provided $\sum_{i=1}^{\infty} p_i x_{i3}$ exists

while if X is continuous assuming values in

(a , b) or [a , b) or (a , b] or [a , b] , finite or infinite a & b,

with probability density function f(x) then

$$E_c(X) = \left\{\int_a^b x^3 f(x)dx\right\}^{\frac{1}{3}}$$

Cubic Expectation of Function of Random Variable

Let

$$\varphi(.) = \varphi(X)$$

be a real valued function of X.

If X is finite discrete then the cubic expectation of φ(X), denoted by $E_c\{\varphi(X)\}$, is defined by

$$E_c\{\varphi(X)\} = \left[\sum_{i=1}^{N} p_i\{\phi(x_i)\}^3\right]^{\frac{1}{3}}$$

and if X is denumerable discrete then

$$E_c\{\varphi(X)\} = \left[\sum_{i=1}^{\infty} p_i\{\phi(x_i)\}^3\right]^{\frac{1}{3}}$$

provided $\sum_{i=1}^{\infty} p_i\{\phi(x_i)\}^3$ exists

while if X is continuous then

$$E_c\{\varphi(X)\} = \left[\int_a^b \{\varphi(x_i)\}^3 f(x_i)\right]^{\frac{1}{3}}$$

Note:

Cubic expectation of a random variable X is the cube root of the arithmetic expectation of its cube i.e. of X 3 and also the cubic expectation of a function φ (X) of X is the cube root of the arithmetic expectation of the cube of φ (X) i.e. of $\{\phi(X)\}^3$.

3. CUBIC EXPECTATION OF CUBIC MEAN

Let us abbreviate arithmetic mean, cubic mean, arithmetic expectation & cubic expectation by AM, CM, AE & CE respectively.

In the Case of Random Variable

Suppose,

$$X_1 , X_2 , \text{.........} , X_k$$

are k real valued random variables.

Then from the definitions of E_A(X) and E_C(X),

$$\{E_C (X_i\}3 = EA(X_i 3) \quad \text{or} \quad E_C (X_i) = \{ E_A(X_i)\}1/3 \quad , \quad i = 1 , 2 , \text{.......} , k$$

Replacing X_i by $X_i 1/3$, it is obtained that

$$\{E_C (X_i 1/3\}3 = E_A(X_i) \quad \text{or} \quad E_C (X_i 1/3) = \{ E_A(X_i)\}1/3 \quad , \quad i = 1 , 2 , \text{.......} , k$$

Accordingly,

$$[E_C\{(\textstyle\sum_{i=1}^{k} X_i)1/3\}]3 = E_A(\textstyle\sum_{i=1}^{k} X_i)5$$

By additive property of arithmetic expectation,

$$E_A(\textstyle\sum_{i=1}^{k} X_i) = \textstyle\sum_{i=1}^{k} E_A(X_i)$$

Accordingly,

$$[E_C \{(\textstyle\sum_{i=1}^{k} X_i)1/3\}]3 = \textstyle\sum_{i=1}^{k} \{E_C (X_i 1/3)\}3$$

Now, replacing X_i by $X_i 3$ it is obtained that

$$[E_C \{(\textstyle\sum_{i=1}^{k} X_i 3)1/3\}]3 = \textstyle\sum_{i=1}^{k} \{E_C (X_i)\}3$$

which implies,

$$E_C \{(\frac{1}{k} \textstyle\sum_{i=1}^{k} X_i 3)1/3\} = [\frac{1}{k} \textstyle\sum_{i=1}^{k} \{E_C (X_i)\}3]1/3$$

This means,

$$E_C \{CM (X_1 , X_2 , \text{.........} , X_k)\} = CM\{ E_C (X_1) , E_C (X_2) , \text{........} , E_C (X_k)\}$$

In the Case of Function of Random Variable

Let

$$\phi 1(X_1) = \phi 1 \,,\, \phi 2(X_1) = \phi 2 \,,\, \ldots\ldots\ldots \,,\, \phi k(X_k) = \phi k$$

be k functions of

$$X_1 , X_2 , \ldots\ldots\ldots , X_k$$

Respectively.

Then from the definitions of EA{ φ (X)},

$$\{E_C (\phi\, i\}3 = E_A(\phi\, i\, 3) \quad \text{or} \quad E_C (\phi\, i) = \{E_A (\phi\, i)\}1/3 \quad , \quad i = 1 , 2 , \ldots\ldots , k$$

Proceeding with the same logic, applied in the above case, it can be obtained that

$$E_C \{(\frac{1}{k} \sum_{i=1}^{k} \phi\, i\, 3)1/3\} = [\frac{1}{k} \sum_{i=1}^{k} \{E_C (\phi\, i)\}3]1/3$$

i.e.

$$E_C \{CM (\phi 1 , \phi\, 2 , \ldots\ldots\ldots , \phi k)\} = CM\{ EC (\phi 1) , E_C (\phi\, 2) , \ldots\ldots\ldots , E_C (\phi k)\}$$

4. CONCLUSION

Property of cubic expectation, obtained above, can be summarized as follows:

"Cubic expectation of cubic mean of a finite number of random variables is the cubic mean of the individual cubic expectations of the variables and cubic expectation of cubic mean of a finite number of individual functions of respective random variables is the cubic mean of the individual cubic expectations of the respective functions."

This property of cubic expectation is similar to those of arithmetic expectation, geometric expectation, harmonic expectation and quadratic expectations which had already been established [9 , 11 , 12].

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