Partners Universal International Innovation Journal (PUIIJ)
Volume: 03 Issue: 06 | November-December 2025 | ISSN: 2583-9675 | www.puiij.com

Quadratic Expectation of Quadratic Mean of Random Variables

Dhritikesh Chakrabarty

Independent Researcher, Guwahati, Assam, India.

Abstract - Concepts of quadratic expectation of random variable was introduced in a study with formulating its definition and deriving some elementary properties of itself. It has been thought that quadratic expectation might carry some more properties which are still unknown to us. One of its possible unknown properties has been identified and derived in this study which has been presented here.

Keywords: Random Variable, Quadratic expectation, Quadratic Mean, Property.

1. INTRODUCTION

Expectation, a statistical concept associated to random variable, is the theoretical average [1 , 12] of the possible values assumed by the variable [3, 5 , 13 , 14 , 17 , 18 , 27]. It is defined as the weighted average of its all possible values with their respective probabilities as the corresponding weights [3 , 13 , 14 , 17 ,18 , 23]. Originally, expectation had been defined as the weighted arithmetic mean [2 , 4 , 20] of its all possible values with their respective probabilities as the corresponding weights and was termed as mathematical expectation [3 , 13 , 14 , 17 ,18 , 23] which was termed as arithmetic expectation in later studies [6 , 11]. In some other studies, three more concepts of expectation had been introduced and defined using the concepts of geometric mean [2 , 4], harmonic mean [2 , 4 , 21] and quadratic mean [15 , 16 , 20] which were termed as geometric expectation [6 , 7 , 11], harmonic expectation [6 , 8 , 9 , 11] and quadratic expectation [10 , 11] respectively.

Each of the four concepts/definitions of expectation carries its own properties which may be known or unknown to us. Some properties of arithmetic expectation had already been derived and are available in the literature of statistics [3 , 13 , 14 , 17 ,19 , 23]. Similarly, some properties of each of were geometric expectation, harmonic expectation and quadratic expectation were also been derived in some studies [7 , 8 , 9, 10]. It has been thought that quadratic expectation might carry some more properties which are still unknown to us. One of its possible unknown properties has been identified and derived in this study which has been presented here.

2. QUADRATIC EXPECTATION

Let us consider a random variable denoted by X.
If the random variable X is discrete and assumes real values

$$x_1 , x_2 , \dots\dots\dots , x_N$$

with respective probabilities

$$p_1 , p_2 , \dots\dots\dots , p_N$$

then the quadratic expectation of X, denoted by $E_Q(X)$, is defined by

$$E_Q(X) = \sqrt{\sum_{i=1}^{N} p_i x_i^2}$$

where absolute value of square root is taken [10].

If X is a continuous and assumes real values in the interval

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then quadratic expectation of X denoted by $E_Q(X)$ can be defined by

$$E_Q(X) = \sqrt{\int_a^b x^2 . f(x) dx}$$

where absolute value of square root is taken [11].

Some Notable Points:

Note (1):

Quadratic expectation describes the theoretical absolute magnitude of a random variable.

Note (2):

Arithmetic expectation of the random variable, denoted by $E_A(X)$, is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i \quad ,$$

if X is discrete [6]

and

$$E_A(X) = \int_a^b x . f(x) dx \quad ,$$

if X is continuous [10]

which implies,

$$E_A(X^2) = \{E_Q(X)\}^2 \quad \text{or.} \quad E_Q(X) = \sqrt{E_A(X^2)}$$

This means, quadratic expectation of a random variable X can also be defined as the absolute square root of arithmetic expectation of its square (i.e. of X^2).

Note (3):

One consequence of Note (2) is that

$$\{E_Q(\sqrt{X})\}^2 = \{E_A(X) \quad \text{or} \quad E_Q(\sqrt{X}) = \sqrt{E_A(X)}$$

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3. QUADRATIC EXPECTATION OF QUADRATIC MEAN

Let us abbreviate arithmetic mean, quadratic mean, arithmetic expectation & quadratic expectation by AM, QM, AE & QE respectively.

To obtain the relation between quadratic expectation of quadratic mean of a number of random variables let us first prove the following theorem namely Theorem (1):

Theorem (1):

If

$$X_1 , X_2 , \,\ldots\ldots\ldots , X_k$$

are k real valued random variables then

$$\{ E_Q (\sqrt{X_1 + X_2 + \ldots\ldots\ldots + X_k}\,)\}^2 = \{ E_Q (\sqrt{X_1}\,)\}^2 + \{ E_Q (\sqrt{X_2}\,)\}^2 + \ldots\ldots$$
$$\ldots\ldots + \{ E_Q (\sqrt{X_k}\,)\}^2$$

Proof:

From Note (3), it is obtained that

$\{E_Q (\sqrt{X_1})\}^2 = E_A (X_1)$ i.e. $E_Q (\sqrt{X_1}) = \sqrt{E_A(X_1)}$,

$\{E_Q (\sqrt{X_2})\}^2 = E_A (X_2)$ i.e. $E_Q (\sqrt{X_2}) = \sqrt{E_A(X_2)}$,

……………………………………………………… ,

$\{E_Q (\sqrt{X_k})\}^2 = EA (X_k)$ i.e. $E_Q (\sqrt{X_k}) = \sqrt{E_A(X_k)}$,

$\{E_Q (\sqrt{X_1 + X_2 + \ldots\ldots\ldots + X_k}\,)\}^2 = E_A(X_1 + X_2 + \ldots\ldots\ldots + X_k)$

i.e. $E_Q (\sqrt{X_1 + X_2 + \ldots\ldots\ldots + X_k}\,) = \sqrt{E_A(X_1 + X_2 + \ldots\ldots\ldots + X_k)}$

By additive property of arithmetic expectation [3 , 10 – 12 , 22 , 23],

$$E_A (X_1 + X_2 + \ldots\ldots\ldots + X_k) = E_A (X_1) + E_A (X_2) + \ldots\ldots\ldots + E_A (X_k)$$

Therefore,

$$\{ E_Q (\sqrt{X_1 + X_2 + \ldots\ldots\ldots + X_k}\,)\}^2 = \{E_Q (\sqrt{X_1}\,)\}^2 + \{E_Q (\sqrt{X_2}\,)\}^2 + \ldots\ldots$$
$$\ldots\ldots + \{E_Q (\sqrt{X_k}\,)\}^2$$

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This can be regarded as additive property of quadratic expectation.

The relation between quadratic expectation of quadratic mean of a number of random variables can be stated as in the following theorem namely Theorem (2):

Theorem (2):

QE of QM of a number of random variables is the QM of the QEs of the squares of the variables i.e. if

$$X_1 , X_2 , \text{..........} , X_k$$

are k random variables then

$E_Q\{\text{QM}(X_1 , X_2 , \text{.........} , X_k)\} = \text{QM}\{E_Q(X_1) , E_Q(X_2) , \text{........} , E_Q(X_k)\}$

Proof: From Property (1), it is obtained that

$$\left\{E_Q(\sqrt{X_1^2 + X_2 + \text{..........} + X_k^2})\right\}^2$$

$= \{E_Q(X_1)\}^2 + \{E_Q(X_2)\}^2 + \text{.........} + \{E_Q(X_k)\}^2$

This implies,

$$E_Q\left(\sqrt{\frac{1}{k}(X_1^2 + X_2^2 + \text{.........} + X_k^2)}\right)$$

$= \sqrt{\frac{1}{k}[\{E_Q(X_1)\}^2 + \{E_Q(X_2)\}^2 + \text{..........} + \{E_Q(X_k)\}^2]}$

This means,

$E_Q\{\text{QM }(X_1 , X_2 , \text{.........} , X_k)\} = \text{QM}\{E_Q(X_1) , E_Q(X_2) , \text{........} , E_Q(X_k)\}$

4. CONCLUSION

Additive property of arithmetic expectation namely

$$E_A(X_1 + X_2 + \text{..........} + X_k) = E_A(X_1) + E_A(X_2) + \text{..........} + E_A(X_k)$$

implies that

$$E_A\{\frac{1}{k}(X_1 + X_2 + \text{.........} + X_k)\} = \frac{1}{k}\{E_A(X_1) + E_A(X_2) + \text{..........} + \{E_Q(X_k)\}$$

This means,

$$E_A\{\text{AM }(X_1 , X_2 , \text{.........} , X_k)\} = \text{AM}\{E_A(X_1) , E_A(X_2) , \text{........} , E_A(X_k)\}$$

i.e. the AE of AM of a number of random variables is the AM of the individual AEs of the variables.

The property of QE derived here is similar to this property of AE.

In this connection, it is to be mentioned that it is still not known whether geometric expectation and harmonic expectation carry some properties like that of arithmetic expectation and quadratic expectation. This is a problem of research at this stage.

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