Partners Universal International Innovation Journal (PUIIJ)
Volume: 03 Issue: 05 | September-October 2025 | ISSN: 2583-9675 | www.puiij.com

Geometric Unbiased Estimator: Some Properties

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Abstract - Concepts of geometric unbiased estimator was introduced in a recent study. Attempt has here been made on identifying some important properties/facts/results of geometric unbiased estimator. This article is based on the information on this unbiased estimator obtained in the attempt.

Keywords: Estimation, Geometric Unbiasedness Estimator, Some Properties.

1. INTRODUCTION

In the theory of statistical estimation, unbiasedness is regarded as a desirable quality/quality of an estimator [1 , 8 , 12, 13]. Originally, the concept of unbiasedness [8 , 9] was explained on the basis of the mathematical expectation [2 , 11 , 14], specifically the arithmetic expectation, of the estimator concerned and accordingly unbiased estimator was defined [1 , 12 , 13]. This definition later was termed as arithmetic unbiased estimator [6]. Recently, concepts of geometric unbiased estimator [6] was introduced and defined on the basis of geometric expectation [3 , 4 , 5 , 7]. Attempt has here been made to identify some important properties/facts/results of geometric unbiased estimator. This article is based on the information on this unbiased estimator obtained in the attempt.

2. GEOMETRIC UNBIASED ESTIMATOR

Suppose

$$X_1 , X_2 , \ldots\ldots\ldots , X_n$$

is a random sample drawn from a population which follows a probability distribution having parameter θ

$$\& \qquad T = T(X_1 , X_2 , \ldots\ldots\ldots , X_n)$$

is an estimator of θ.

T can be regarded as geometric unbiased estimator [6] of parameter θ if

$$E_G(T) = \theta$$

where EG(T) is the geometric expectation of T.

Note:

(1) Geometric unbiased estimator can exist in the case of only strictly positive valued estimator but not for any real valued estimator.

It is to be noted that the value of θ, in this case, is an unknown positive real number.

(2) Geometric expectation of a random variable can also be defined as the antilogarithm of the arithmetic expectation of its logarithm.

Accordingly, $E_G(T)$ can also be defined as the antilogarithm of the arithmetic expectation of the logarithm of T i.e.

$$E_G(T) = \text{Antilog}\{E_A(\log T)\}$$

3. ALGEBRAIC PROPERTIES OF GEOMETRIC UNBIASED IN ESTIMATOR

Property (1): There may exists more than one geometric unbiased estimator of a parameter.

Proof: Let us consider a population following the uniform discrete distribution [10] described by the probability mass function

$$P(X = x_i) = \frac{1}{K} \quad , \quad (x_i = 1, 2, \ldots\ldots, K)$$

with population geometric mean μ_Q where

$$\mu_G = \left(\prod_{r=1}^{K} r \right)^{1/K}$$

Suppose

$$X_1, X_2, \ldots\ldots, X_n$$

is a random sample drawn from this population.

Then

$$E_G(X_i) = \left(\prod_{r=1}^{K} r\right)^{1/K} = \mu_G \ , \text{ for each } X_i \quad (i = 1, 2, \ldots\ldots, k)$$

This implies each X_i is geometric unbiased estimator of parameter μ_G.

Similarly, each of

$$\bar{X}_G(2), \bar{X}_G(3), \ldots\ldots\ldots, \bar{X}_G(n)$$

where

$$\bar{X}_G(r) = \left(\prod_{i=1}^{r} X_i \right)^{1/r}$$

is also geometric unbiased estimator of parameter μ_G

since

$$E_G\{\bar{X}_G(r)\} = E_G\left(\prod_{i=1}^{r} X_i \right)^{1/r} = \left\{ \prod_{i=1}^{r} E_G(X_i) \right\}^{1/r} = \mu_G$$

Thus, Property (1) has been obtained:

Property (2): There may not exists geometric unbiased estimator of a parameter.

Proof: Let us consider a population following binomial distribution [10] having parameters R (number of trials) and p (probability of success).

Suppose

$$X_1\ ,X_2\ ,\ \ldots\ldots\ ,X_n$$

is a random sample drawn from this population.

For this distribution, geometric unbiased estimator of the binomial parameter p does not exist.

Thus, the following property has been obtained:

Thus, Property (2) has been obtained:

Property (3): If T is geometric unbiased estimator of parameter θ then Tm is geometric unbiased estimator of parameter θ^{m}.

Proof: Let T be a geometric unbiased estimator of parameter θ.

Then

$$E_G(T) = \theta$$

By index property of geometric expectation [5],

$$E_G(T^m) = \{E_G(T)\}^m = \theta^m$$

i.e. Tm is geometric unbiased estimator of parameter θ m.

Hence, Property (3) has been obtained:

Property (4): Geometric mean of a finite number of geometric unbiased estimators of a parameter is also geometric unbiased estimator of the parameter.

Proof:

Let T & S be two geometric unbiased estimators of parameter θ.

Then

$$EG(T) = \theta \quad \& \quad EG(S) = \theta$$

By multiplicative property of geometric expectation [5],

$$E_G(TS) = E_G(T)\ E_G(S) = \theta^2$$

$$\text{i.e.} \quad \theta = \{ E_G(T)\ E_G(S)\}^{1/2} = \{E_G(TS)\}^{1/2} = E_G\ \{(TS)^{½}\}$$

i.e. $(TS)^{½}$ is geometric unbiased estimator of parameter θ.

Therefore, if T & S are two geometric unbiased estimators of parameter θ then $(TS)^{½}$ i.e. the simple geometric mean of T & S is also geometric unbiased estimator of parameter θ.

Now, let

$$T_1\ , T_2\ ,\ \ldots\ldots\ , T_k$$

be k geometric unbiased estimators of parameter θ.

Then

$$E_G(T_1) = \theta \;,\; E_G(T_2) = \theta \;,\; \ldots\ldots\ldots \;,\; E_G(T_k) = \theta$$

By multiplicative property of geometric expectation [5],

$$E_G(T_1 \,.\, T_2 \,.\, \ldots\ldots\ldots \,.\, T_k) = E_G(T_1) \,.\, E_G(T_2) \,.\, \ldots\ldots\ldots \,.\, E_G(T_k) = \theta\, k$$

$$\text{i.e.}\quad \{E_G(T_1) \,.\, E_G\,\{T_2) \,.\, \ldots\ldots\ldots \,.\, E_G(T_k)\}1/k \;= \theta$$

$$\text{i.e.}\quad \{EG(T_1 \,.\, T_2 \,.\, \ldots\ldots\ldots \,.\, T_k)\; 1/k\;\} = \theta$$

then

$$(T_1.T_2.\; \ldots\ldots\ldots \;.T_k\,)1/k$$

i.e. the simple geometric mean of

$$T_1 \,, T_2 \,,\; \ldots\ldots\ldots \;, T_k$$

is geometric unbiased estimator of parameter θ.

Again, let

$$T_1 \,, T_2 \,,\; \ldots\ldots\ldots \;, T_k$$

be k geometric unbiased estimators of parameter θ

&

$$w_1 \,, w_2 \,,\; \ldots\ldots\ldots \;, w_k$$

are k positive real numbers such that .

$$w = w_1 + w_2 + \;\ldots\ldots\ldots\; + w_k$$

Then

$$E_G\,\{(T_1{}^{w_1}.T_2{}^{w_2}.\;\ldots\ldots\ldots\;.T_k{}^{w_k}\,)^{\frac{1}{w}}\,\}$$

$$= \{E_G\,(T_1{}^{w_1}.T_2{}^{w_2}.\;\ldots\ldots\ldots\;.T_k{}^{w_k}\,)\}^{\frac{1}{w}}$$

$$= \{E_G\,(T_1{}^{w_1}) \,.\, EG\,(T_2{}^{w_2}).\;\ldots\ldots\ldots\ldots\;.\, E_G\,(T_k{}^{w_k})\,\}^{\frac{1}{w}}$$

$$= (\theta^{w_1} \,.\, \theta^{w_2} \,.\;\ldots\ldots\ldots\ldots\;.\, \theta^{w_k}\,)^{\frac{1}{w}}$$

$$= \theta$$

Therefore

$$(T_1{}^{w_1}.T_2{}^{w_2}.\;\ldots\ldots\ldots\;.T_k{}^{w_k}\,)^{\frac{1}{w}}$$

where

$$w = w_1 + w_2 + \;\ldots\ldots\ldots\; + w_k$$

i.e. the weighted geometric mean of

$$T_1 \,, T_2 \,,\; \ldots\ldots\ldots \;, T_k$$

with

$$w_1 , w_2 , \text{ ……… } , w_k$$

as the respective weights of

$$T_1 , T_2 , \text{ ……… } , T_k$$

is geometric unbiased estimator of parameter θ.

Hence, Property (4) has been obtained:

Property (5): If $S_1 , S_2 ,$ ……… $, S_m$ are geometric unbiased estimators of the respective parameters $\theta_1 , \theta_2 ,$ ……… $, \theta_m$, then $(S_1 . . S_2 .$ ……… $. S_m)^{\frac{1}{m}}$

is geometric unbiased estimator of $(\theta_1 . \theta_2 .$ ……… $.\theta_m)^{\frac{1}{m}}$.

Proof: Suppose

$$S_1 , S_2 , \text{ ……… } , S_m$$

are geometric unbiased estimators of parameters

$$\theta_1 , \theta_2 , \text{ ……… } , \theta_m$$

respectively.

Then

$$E_G(S_1) = \theta_1 \quad , \quad E_G(S_2) = \theta_2 \quad , \text{ …………… }, \ E_G(S_m) = \theta_m$$

Now

$$E_G(S_1 . S_2 . \text{ … … … } . S_m) = E_G(S_1). E_G(S_2). \text{ ……… }. E_G(S_m)$$

$$\text{i.e.} \quad EG(S_1 . S_2 . \text{ … … … } . S_m) = \theta_1 . \theta_2 . \text{ ……… } .\theta_m$$

which also implies

$$E_G\{(S_1 . . S_2 . \text{ ……… } . S_m)^{\frac{1}{m}}\} = \{EG(S_1 . . S_2 . \text{ ……… } . S_m)\}^{\frac{1}{m}}$$

$$\text{i.e.} \quad E_G\{(S_1 . . S_2 . \text{ ……… } . S_m)^{\frac{1}{m}}\} = (\theta_1 . \theta_2 . \text{ ……… } .\theta_m)^{\frac{1}{m}}$$

Therefore,

$$S_1 . . S_2 . \text{ ……… } . S_m$$

is geometric unbiased estimator of

$$\theta_1 . \theta_2 . \text{ ……… } .\theta_m$$

and

$$(S_1 . . S_2 . \text{ ……… } . S_m)^{\frac{1}{m}}$$

is geometric unbiased estimator of

$$(\theta_1 . \theta_2 . \text{ ……… } .\theta_m)^{\frac{1}{m}}$$

Hence, Property (5) has been obtained:

Property (6): If

$$S_1 , S_2 , \dots\dots\dots , S_m , S_{m+1} , S_{m+2} , \dots\dots\dots , S_p$$

are geometric unbiased estimators of the respective parameters

$$\theta_1 , \theta_2 , \dots\dots\dots , \theta_m , \theta_{m+1} , \theta_{m+2} , \dots\dots\dots , \theta_p$$

then

$$\frac{S_1 . S_2 . \dots\dots\dots . S_m}{S_{m+1} . S_{m+2} . \dots\dots\dots . S_p}$$

is geometric unbiased estimator of

$$\frac{\theta_1 . \theta_2 . \dots\dots\dots . \theta_m}{\theta_{m+1} . \theta_{m+2} . \dots\dots\dots . \theta_p}$$

Proof: In this case,

$E_G(S_1) = \theta_1$, $E_G(S_2) = \theta_2$, ..., $E_G(S_m) = \theta_m$, $E_G(S_{m+1}) = \theta_{m+1}$, $E_G(S_{m+2}) = \theta_{m+2}$, ... , $E_G(S_p) = \theta_p$

$$S_{m+1} S_1 , S_2 , \dots\dots\dots , S_m , S_{m+1} , S_{m+2} , \dots\dots\dots , S_p$$

are geometric unbiased estimators of the respective parameters

$$\theta_1 , \theta_2 , \dots\dots\dots , \theta_m , \theta_{m+1} , \theta_{m+2} , \dots\dots\dots , \theta_p$$

respectively.

First applying the quotient property & then the multiplicative property of geometric expectation [ ], it is obtained that

$$E_G\left\{ \frac{S_1 . S_2 . \dots\dots\dots . S_m}{S_{m+1} . S_{m+2} . \dots\dots\dots . S_p} \right\} = \frac{\theta_1 . \theta_2 . \dots\dots\dots . \theta_m}{\theta_{m+1} . \theta_{m+2} . \dots\dots\dots . \theta_p}$$

Therefore,

$$\frac{S_1 . S_2 . \dots\dots\dots . S_m}{S_{m+1} . S_{m+2} . \dots\dots\dots . S_p}$$

is geometric unbiased estimator of

$$\frac{\theta_1 . \theta_2 . \dots\dots\dots . \theta_m}{\theta_{m+1} . \theta_{m+2} . \dots\dots\dots . \theta_p}$$

Thus, Property (6) is obtained:

4. CONCLUSION

Concept of geometric unbiasedness is likely to be useful and/or helpful in finding unbiased estimator of a parameter in the situation where the associated data are of ratio type or other allied types.

The properties of geometric unbiased estimator, obtained here, are likely to be useful and/or helpful in finding unbiased estimator of a function of parameter in the similar situations.

Moreover, the properties of geometric unbiased estimator are likely to be important and useful in enriching the theory of statistical estimation.

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