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Arithmetic, Geometric, Harmonic and Quadratic Unbiasedness in Estimation

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Abstract - Unbiasedness of estimator has here been defined on the basis of geometric expectation, harmonic expectation & quadratic expectation which are based on the concept of geometric mean, harmonic mean and quadratic mean respectively. Unbiased estimators, defined on the basis of these three, have been termed here as geometric unbiased, harmonic unbiased & quadratic unbiased respectively.

Keywords: Estimator, Unbiasedness, Arithmetic, Geometric, Harmonic, Quadratic.

1. INTRODUCTION

Unbiasedness, which in general refers to the quality of being fair and impartial, without prejudice or favoritism, can be applied to various situations, from opinions and judgments to statistical estimation [1 , 14].

In statistics, an estimator of a parameter is defined to be a method for calculating a parameter's value from sample data [2 , 11] and the concept of unbiasedness which is regarded as a quality of estimator is defined on the basis of the expectation (or expected value) of the estimator [3 , 12 , 15 , 17 , 20]. Unbiasedness means that the expected value of an estimator is equal to the true value of the parameter being estimated [1 , 12 , 20]. This means that, on average, the estimator will produce the correct value over many trials.

In statistics, bias of an estimator is the difference between this estimator's expected value and the true value of the parameter being estimated [13 , 18 , 19] and accordingly an estimator with zero bias is called unbiased.

In probability theory, the expected value (also called expectation, expectancy, and mathematical expectation) of a random variable is defined in general as weighted average of the possible values a random variable can take, weighted by the respective probabilities of those possible values [3 ,10 , 12 , 15 , 17 , 20]. However, the definition of expectation had been existing for a long time on the basis of arithmetic mean only and thus this definition of expectation can more specifically be termed as arithmetic expectation [4 , 5]. Recently, expectation has been defined on the basis of the concept of geometric mean, harmonic mean and quadratic mean which have been termed as geometric expectation [5 , 6 , 7], harmonic expectation [4 , 5 , 7 , 8] & quadratic expectation [9] respectively.

The existing definition of unbiased estimator is based on the concept of arithmetic expectation [12 , 15 , 17 , 20]. Since unbiased estimator of a parameter is that which estimates the parameter where the expected value of the estimator equals the true value of the parameter being estimated, it can also be defined on the basis of geometric expectation, harmonic expectation & quadratic expectation. Unbiasedness of estimator has here been defined on the basis of these three types of expectation. Unbiased estimators, defined on the basis of these three types of expectation namely geometric expectation, harmonic expectation & quadratic expectation have been termed here as geometric unbiased estimator, harmonic unbiased estimator & quadratic unbiased estimator respectively.

2. FOUR TYPES OF UNBIASED ESTIMATOR

Let

$$X_1 , X_2 , \ldots\ldots , X_n$$

be a random sample drawn from a population having parameter θ

$$\& \qquad T = T(X_1 , X_2 , \ldots\ldots , X_n)$$

be an estimator of the parameter θ.

If θ0 is the true value of the parameter θ, then B(T) given by

$$B(T) = E(T) - \theta_0$$

where E(T) is the expected value of T

is the bias of estimator T of parameter θ.

$$\text{If} \quad B(T) = 0 \quad \text{i.e.} \quad E(T) = \theta_0 ,$$

then T is unbiased estimator of parameter θ.

Four types of unbiased estimators can be defined using the four definitions of expectation namely arithmetic expectation, geometric expectation, harmonic expectation and quadratic expectation.

Arithmetic Unbiased Estimator:

Applying the concept/definition of arithmetic expectation in the general concept/definition of unbiased estimator as mentioned above, it is obtained that

T can be regarded as arithmetic unbiased estimator of parameter θ if

$$E_A(T) = \theta_0$$

where $E_A(T)$ is the arithmetic expectation of T.

Bias B_A (T) of estimator T for parameter θ, in this case, becomes

$$B_A(T) = E_A(T) - \theta_0$$

which can be regarded as arithmetic bias of estimator T for parameter θ.

E_A (X) can exist and can be defined for any real valued random variable X [4 , 5].

Accordingly, $E_A(T)$ can exists for any real valued estimator T of parameter θ.

This means, arithmetic unbiased estimator exist in the case of any real valued estimator.

It is to be noted that the value of parameter θ, in this case, is an unknown real number.

In this connection, it is to be mentioned that this is the concept/definition of unbiasedness usually known/used and are available in the existing literature of statistics.

Geometric Unbiased Estimator:

If the concept/definition of geometric expectation is applied in the general concept/definition of unbiased estimator, it is obtained that

T can be regarded as geometric unbiased estimator of parameter θ if

$$E_G(T) = \theta_0$$

where $E_G(T)$ is the geometric expectation of T.

Bias $B_G(T)$ of estimator T for parameter θ, in this case, becomes

$$B_G(T) = E_G(T) - \theta_0$$

which can be regarded as geometric bias of estimator T for parameter θ.

$E_G(X)$ can exist and can be defined for strictly positive valued random variable X [4 , 5 , 6 but not for any real valued random variable X.

Accordingly, $E_G(T)$ can exist for strictly positive valued estimator T of parameter θ.

This means, geometric unbiased estimator can exist in the case of only strictly positive valued estimator but not for any real valued estimator.

It is to be noted that the value of θ, in this case, is an unknown positive real number.

Harmonic Unbiased Estimator:

In a similar manner, if the concept/definition of harmonic expectation is applied in the general concept/definition of unbiased estimator, it is obtained that

T can be regarded as harmonic unbiased estimator of parameter θ if

$$E_H(T) = \theta_0$$

where $E_H(T)$ is the harmonic expectation of T.

Bias $B_H(T)$ of estimator T for parameter θ, in this case, becomes

$$B_H(T) = E_H(T) - \theta_0$$

which can be regarded as harmonic bias of estimator T for parameter θ.

$E_H(X)$ can exist and can be defined for non-zero real valued random variable X [4 , 5 , 7 , 8].

Accordingly, $E_H(T)$ can exist for non-zero real valued estimator T of parameter θ.

This means, harmonic unbiased estimator can exist in the case of non-zero real valued estimator.

It is to be noted that the value of θ, in this case, is an unknown non-zero real number.

Quadratic Unbiased Estimator:

Again, if the concept/definition of quadratic expectation is applied in the general concept/definition of unbiased estimator, it is obtained that

T can be regarded as quadratic unbiased estimator of parameter θ if

$$E_Q(T) = \theta_0$$

where $E_Q(T)$ is the harmonic expectation of T.

Bias $B_Q(T)$ of estimator T for parameter θ, in this case, becomes

$$B_Q(T) = E_Q(T) - \theta 0$$

which can be regarded as quadratic bias of estimator T for parameter θ.

$E_Q(X)$ can exist and can be defined for any real valued random variable X [9].

Accordingly, $E_Q(T)$ can exist for any real valued estimator T of parameter θ.

This means, quadratic unbiased estimator exist in the case of any real valued estimator.

It is to be noted that the value of parameter θ, in this case, is an unknown real number.

Note:

(1) Geometric expectation of a random variable can also be defined as the antilogarithm of the arithmetic expectation of its logarithm [4].

Accordingly, $E_G(T)$ can also be defined as the antilogarithm of the arithmetic expectation of the logarithm of T i.e.

$$E_G(T) = \text{Antilog}\ \{E_A(\log T)\}$$

Similarly, harmonic expectation of a random variable can also be defined as the reciprocal of the arithmetic expectation of its reciprocal [4].

Accordingly, EH(T) can also be defined as the reciprocal of the arithmetic expectation of the reciprocal of T i.e.

$$E_H(T) = \{E_A(T -1)\} -1$$

Again, quadratic expectation of a random variable can also be defined as the square root of the arithmetic expectation of its square [9].

Accordingly, $E_Q(T)$ can also be defined as the square root of the arithmetic expectation of the square of T i.e.

$$EQ(T) = \{EA(T\ 2)\}1/2$$

(2) All of $E_A(X)$, $E_G(X)$, $E_H(X)$ & $E_Q(X)$ can exist for strictly positive valued random variable X and satisfy the inequality [9]

$$E_A(X) > E_G(X) > E_H(X) > E_Q(X)$$

Accordingly, all of $E_A(T)$, $E_G(T)$, $E_H(T)$ & $E_Q(T)$ can exist for strictly positive valued estimator T and satisfy the inequality

$$E_A(T) > E_G(T) > E_H(T) > E_Q(T)$$

This means, if one of these four expectations is equal to θ, the other three expectations cannot be equal to θ.

This implies, if T is one type of unbiased estimator of parameter θ, it cannot be any other type of unbiased estimator of θ. For example, if T is arithmetic unbiased estimator of parameter θ, it cannot be any other unbiased (i.e. geometric unbiased, harmonic unbiased & quadratic unbiased) estimator of θ.

3. UNBIASED ESTIMATOR OF FUNCTION OF PARAMETER

The above concepts/definitions of arithmetic unbiased, geometric unbiased, harmonic unbiased & quadratic unbiased can be generalized to the case of function of parameter.

Let

$$S = S(X_1, X_2, \ldots\ldots, X_n)$$

be an estimator of $\phi(\theta)$, a function of parameter θ.

Suppose $\phi 0 = \phi(\theta 0)$ is the true value of $\phi(\theta)$.

Arithmetic Unbiased Estimator of $\phi(\theta)$:

S can be regarded as arithmetic unbiased estimator of $\phi(\theta)$ if

$$E_A(S) = \phi 0$$

where $\phi 0$ & S are both real valued.

Bias $B_A(S)$ of estimator S of $\phi(\theta)$, in this case, becomes

$$B_A(S) = E_A(S) - \phi 0$$

which can be regarded as arithmetic bias of estimator S of $\phi(\theta)$.

Geometric Unbiased Estimator of $\phi(\theta)$:

S can be regarded as geometric unbiased estimator of $\phi(\theta)$ if

$$E_G(S) = \phi 0$$

provided S & $\phi 0$ are strictly positive.

Bias $B_G(S)$ of estimator S of $\phi(\theta)$, in this case, becomes

$$B_G(S) = E_G(S) - \phi 0$$

which can be regarded as geometric bias of estimator S of $\phi(\theta)$.

$E_G(S)$ can also be defined as the antilogarithm of the arithmetic expectation of the logarithm of S i.e.

$$E_G(S) = \text{Antilog}\,\{E_A(\log S)\}$$

Harmonic Unbiased Estimator of $\phi(\theta)$:

Similarly, S can be regarded as harmonic unbiased estimator of parameter $\phi(\theta)$ if

$$E_H(S) = \phi 0$$

provided $\phi 0$ & S are both non-zero.

Bias $B_H(S)$ of estimator S of $\phi(\theta)$, in this case, becomes

$$B_H(S) = E_H(S) - \phi 0$$

which can be regarded as harmonic bias of estimator S of $\phi(\theta)$.

$E_H(S)$ can also be defined as the reciprocal of the arithmetic expectation of the reciprocal of S i.e.

$$E_H(S) = \{E_A(S\ {-1})\}\ {-1}$$

Quadratic Unbiased Estimator of $\phi(\theta)$:

Again, S can be regarded as quadratic unbiased estimator of parameter $\phi(\theta)$ if

$$E_H(S) = \phi 0$$

where $\phi 0$ & S are both real valued.

Bias $B_Q(S)$ of estimator S of $\phi(\theta)$, in this case, becomes

$$B_Q(S) = E_Q(S) - \phi 0$$

which can be regarded as quadratic bias of estimator S of $\phi(\theta)$.

$E_Q(S)$ can also be defined as the square root of the arithmetic expectation of the square of S i.e.

$$E_Q(S) = \{E_A(S\ 2)\}1/2$$

Note:

Like the inequality satisfied by $E_A(T)$, $E_G(T)$, $E_H(T)$ & $E_Q(T)$ as mentioned above, in this case also $E_A(S)$, $E_G(S)$, $E_H(S)$ & $E_Q(S)$ can exist for strictly positive valued estimator S and satisfy the inequality

$$E_A(S) > E_G(S) > E_H(S) > E_Q(S)$$

This means, if one of these four expectations is equal to $\phi 0$, the other three expectations cannot be equal to $\phi 0$.

This implies, if S is one type of unbiased estimator of $\phi(\theta)$, it cannot be any other type of unbiased estimator of $\phi(\theta)$. For example, if S is arithmetic unbiased estimator of $\phi(\theta)$, it cannot be any other unbiased (i.e. geometric unbiased, harmonic unbiased & quadratic unbiased) estimator of $\phi(\theta)$.

4. EXAMPLE

Let us consider a population following the discrete uniform distribution [16 , 23 , 24] described by the probability mass function

$$P(X = x_i) = \frac{1}{K} \quad , \quad (x_i = 1, 2, \ldots\ldots, K)$$

Let μ_A , μ_G , μ_H & μ_Q be the population arithmetic mean, the population geometric mean, the population harmonic mean & the population quadratic mean respectively.

Then

$$\mu_A = \frac{1}{K} \sum_{r=1}^{K} r \ ,$$

$$\mu_G = (\prod_{r=1}^{K} r)1/K ,$$

$$\mu_H = \{\frac{1}{K} \sum_{r=1}^{K} r - 1 \}-1$$

$$\& \ \mu_Q = \{\frac{1}{K} \sum_{r=1}^{K} r\ 2 \}1/2 \quad .$$

Suppose,

$$X_1 , X_2 , \ldots\ldots\ldots , X_n$$

is a random sample drawn from this population

and $\overline{X}_A$, $\overline{X}_G$, $\overline{X}_H$ & $\overline{X}_Q$ are the sample arithmetic mean, the sample geometric mean, the sample harmonic mean & the sample quadratic mean respectively.

Then

$\overline{X}_A = \frac{1}{n}\sum_{i=1}^{n} X_i$,

$\overline{X}_G = (\prod_{i=1}^{n} X_i)1/n$,

$\overline{X}_H = \{\frac{1}{n}\sum_{i=1}^{n} X_i - 1 \}-1$

& $\overline{X}_Q = \{\frac{1}{n}\sum_{i=1}^{n} X_i 2\}1/2$

Since each X_i (i = 1 , 2 , ... , n) assumes any one of the K values

1 , 2 , ... , K

with equal probability $\frac{1}{K}$, therefore

$E_A (X_i) = \sum_{r=1}^{K} \frac{1}{K} r = \mu_A$,

$E_G (X_i) = (\prod_{r=1}^{K} r 1/ K) = \mu_G$,

$E_H (X_i) = \{ \sum_{r=1}^{K} \frac{1}{K} r - 1 \}-1 = \mu_H$,

& $E_Q (X_i) = \{ \sum_{r=1}^{K} \frac{1}{K} r 2 \}1/2 = \mu_Q$.

Now

$E_A (\overline{X}_A) = E_A (\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E_A(X_i) = \mu_A$

This implies, $\overline{X}_A$ is arithmetic unbiased estimator of μ_A.

Similarly,

$E_G (\overline{X}_G) = E_G (\prod_{i=1}^{n} X_i)1/n = (\prod_{i=1}^{n} E_G(X_i))1/n = \mu_G$

This implies, $\overline{X}_G$ is geometric unbiased estimator of μ_G .

Again,

$E_H (\overline{X}_H) = \{E_A (\overline{X}_H - 1) \}-1$

$= \{E_A (\frac{1}{n}\sum_{i=1}^{n} X_i - 1) \}-1$

$= \{ \sum_{i=1}^{n} \frac{1}{n} EA (X_i - 1) \}-1$

$= \{ \sum_{i=1}^{n} \frac{1}{n} \mu_H - 1) \}-1$

$= \mu_H$

This implies, $\overline{X}_H$ is harmonic unbiased estimator of μ_H.

Similarly,

$$E_Q(\overline{X}_Q) = \{E_A(\overline{X}_Q2)\}1/2$$
$$= \{E_A(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} X_i2)\}1/2$$
$$= \{\textstyle\sum_{i=1}^{n} \tfrac{1}{n} EA(X_i2)\}1/2$$
$$= \{\textstyle\sum_{i=1}^{n} \tfrac{1}{n}\mu_Q2)\}1/2$$
$$= \mu_Q$$

This implies, $\overline{X}_Q$ is quadratic unbiased estimator of μ_Q.

5. CONCLUSION

The existing definition of unbiased estimator, which is defined in terms of arithmetic expectation, can be regarded as a special definition of unbiased estimator since the concept of unbiasedness is based on the expected value, in general, of the parameter to be estimated. Accordingly, each of the four unbiased estimators defined above is a special type of unbiased estimator.

In reality, data are not of the same type in every situation. Similarly, parameters to be estimated are not of the same characteristic in the case of different datasets. Accordingly, a specific type of unbiasedness may not be valid and proper for finding unbiased estimator of parameter in the case of every dataset. Each of the four concepts of unbiasedness is valid and proper for particular type of parameter to be estimated. For example, if the arithmetic mean of a population is to be estimated from a random sample, then arithmetic unbiasedness is suitable for finding unbiased estimator of the population arithmetic mean. Geometric unbiasedness for estimating the population geometric mean, harmonic unbiasedness for estimating the population harmonic mean and quadratic unbiasedness for estimating the population quadratic mean are suitable for estimating the respective parameters.

Overall, it can be hoped that the concepts of the four types of unbiasedness in estimation will be useful for finding out unbiased estimators of more and more parameters of statistical populations.

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