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Combined Set of Several Sets of Observations: Harmonic Mean

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Abstract - A formula has been derived for computing harmonic mean of a set of observations which is a combined set of a number of sets of observations when the observations are unknown, but the respective harmonic means of the sets and the respective numbers of observations in the sets are known. Derivation of the formula, along with numerical example, has been presented in this article.

Keywords: Observations, Several Sets, Combined Set, Harmonic Mean, Formula.

1. INTRODUCTION

Harmonic Mean which is one of the three basic measures of average namely Arithmetic Mean, Geometric Mean & Harmonic Mean known as Pythagorean means [3 , 6 , 7 , 8 , 16 , 17 , 18] introduced by Pythagoras is not an unused measure of average. In statistics, it is used in measuring central tendency of data [2 , 17 , 18 , 21]. It seems that there are many situations where harmonic mean is suitable to be applied. In a study, it has been used in developing some more measures of central tendency of data namely Arithmetic-Harmonic Mean, Geometric-Harmonic Mean & Arithmetic-Geometric-Harmonic Mean [10 , 11]. Lot of probable properties to be stratified by harmonic mean are yet to be studied. Of course, its additive property has been derived in a recent study [12 , 13]. On the other hand, its probable applications are yet to be identified. There are many situations where harmonic means of several sets of observations are available but the observations are not available and it is required to find out the harmonic mean of all the observations of the sets combined together. This type of situation is common in meta-analysis in clinical trials [1 , 4 , 5 , 14 , 15 , 20]. Attempt has here been made on deriving a formula for computing harmonic mean of a set of observations which is a combined set of a number of sets of observations when the observations are unknown but the respective harmonic means of the sets and the respective numbers of observations in the sets are known. Derivation of the formula, along with numerical example, has been presented in this article.

2. HARMONIC MEAN OF A SET OF NUMBERS

Definition

Let us consider a set of N real numbers or values namely

$$a_1 , a_2 , \ldots\ldots , a_n$$

Harmonic Mean of $a_1 , a_2 , \ldots\ldots , a_n$, denoted by H($a_1 , a_2 , \ldots\ldots , a_n$), is defined by

$$H(a_1 , a_2 , \ldots\ldots , a_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \qquad (2.1)$$

provided the numbers are non-zero [3 , 6 ,7 , 8, 16].

The definition of harmonic mean implies that

$$H(a_1, a_2, \ldots\ldots, a_n) = \frac{n}{\text{Sum of } \frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{a_n}} \qquad (2.2)$$

$$\text{i.e. Sum of } \frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{a_n} = \frac{n}{H(a_1, a_2, \ldots\ldots, a_n)}$$

Thus if h(a) is the harmonic mean of $a_1, a_2, \ldots\ldots, a_n$,
then

$$\text{Sum of } \frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{a_n} = \frac{n}{h(a)} \qquad (2.3)$$

Note:
As per the definition of harmonic mean from first principle [9],
harmonic mean of the numbers

$$a_1, a_2, \ldots\ldots, a_n$$

is the value h(a) such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n} = \frac{1}{h(a)} + \frac{1}{h(a)} + \ldots\ldots + \frac{1}{h(a)}$$

This also implies the equation (2.3).

3. HARMONIC MEAN OF COMBINED SETS OF NUMBERS

Suppose, there are k sets namely

$$S_1, S_2, \ldots\ldots, S_k$$

containing

$$n_1, n_2, \ldots\ldots, n_k$$

observations respectively such that harmonic mean of the sets are

$$h_1, h_2, \ldots\ldots, h_k$$

respectively.

If the observations in the k sets are combined together to form a single set S, then this set will contain

$$n_1 + n_2 + \ldots\ldots + n_k$$

observations.

By equation (2.3),

$$\text{Sum of reciprocals of the observations in the set } S_1 = \frac{n_1}{h_1},$$

$$\text{Sum of reciprocals of the observations in the set } S_2 = \frac{n_2}{h_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{Sum of reciprocals of the observations in the set } S_k = \frac{n_k}{h_k}.$$

This implies,

$$\text{Sum of reciprocals of the observations in the set } S = \frac{n_1}{h_1} + \frac{n_2}{h_2} + \ldots\ldots + \frac{n_k}{h_k}$$

Therefore by equation (2.3), harmonic mean h of the observations in the set S will be

$$h = \frac{n_1 + n_2 + \dots\dots + n_k}{\frac{n_1}{h_1} + \frac{n_2}{h_2} + \dots\dots + \frac{n_k}{h_k}} \qquad (2.4)$$

This is the formula for computing harmonic mean of a combined set of observations from a number of sets of observations when the observations are unknown but the harmonic means of the set and the numbers of observations in the respective sets are known.

This formula has been stated in the form of a theorem below:

Theorem:

If $h_1 , h_2 , \dots\dots , h_k$ are harmonic means of respective sets having $n_1 , n_2 , \dots\dots , n_k$ observations respectively then the harmonic mean of the set of all observations in the k sets combined together is given by

$$h = \frac{n_1 + n_2 + \dots\dots + n_k}{\frac{n_1}{h_1} + \frac{n_2}{h_2} + \dots\dots + \frac{n_k}{h_k}}$$

In particular, if h_x is the harmonic mean of

$$x_1 , x_2 , \dots\dots , x_m$$

& h_y is the harmonic mean of

$$y_1 , y_2 , \dots\dots , y_n$$

Then the harmonic mean h_{xy} of

$$x_1 , x_2 , \dots\dots , x_m , y_1 , y_2 , \dots\dots , y_n$$

is

$$h_{xy} = \frac{m + n}{\frac{m}{h_x} + \frac{n}{h_y}}$$

4. NUMERICAL EXAMPLE

Let us consider the following three sets of numbers

{4 , 8 , 12 , 16 , 20} , {6 , 12 , 18 , 24} & {5 , 10 , 15}

so that

Harmonic Mean of the numbers in the first set = 8.7591240876 ,

Harmonic Mean of the numbers in the second set = 11.5199999999

& Harmonic Mean of the numbers in the first set = 8.1818181818 .

Now, the combined set of the first set and the second set is

{4 , 8 , 12 , 16 , 20 , 6 , 12 , 18 , 24}

so that

Harmonic Mean of the numbers in this set = 9.8033282905

Applying the formula given by equation (2.4), the value of Harmonic Mean of the numbers in the combined set is also found as 9.8033282905 .

Similarly, the combined set of the first set and the third set is

$$\{4 \ , 8 \ , 12 \ , 16 \ , 20 \ , 5 \ , 10 \ , 15\}$$

so that
Harmonic Mean of the numbers in this set = 8.5333333333
Applying the formula given by equation (2.4), the value of Harmonic Mean of the numbers in the combined set is also found as 8.5333333333 .

Also, the combined set of the second set and the third is

$$\{6 \ , 12 \ , 18 \ , 24 \ , 5 \ , 10 \ , 15\}$$

so that

Harmonic Mean of the numbers in this set = 9.8054474708

Applying the formula given by equation (2.4), the value of Harmonic Mean of the numbers in the combined set is also found as 9.8054474708 .

Moreover, the combined set of all the three sets is

$$\{4 \ , 8 \ , 12 \ , 16 \ , 20 \ , 6 \ , 12 \ , 18 \ , 24 \ , 5 \ , 10 \ , 15\}$$

so that
Harmonic Mean of the numbers in this set = 9.3405405405
Applying the formula given by equation (2.4), the value of Harmonic Mean of the numbers in the combined set is also found as 9.3405405405 .

5. CONCLUSION

The type of situation, where respective averages of several sets of observations obtained from independent studies are available but the observations are not available but required to find out the average of all the observations of the sets combined together, is arises in many research studies specially in meta-analysis in clinical trials. The formula derived here is a solution of this type of problem.

In this connection, it is to be mentioned that each of arithmetic mean, geometric mean & harmonic mean is required to be computed in this type of situation. Of course, the formula for computing arithmetic mean in this situation is an established one while the formula for computing harmonic mean has been derived here. However, the formula for computing geometric mean in this situation is yet to be found out. This is a problem of research to be studied at this stage.

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