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Additive Property of Harmonic Expectation From That of Arithmetic Expectation

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Abstract - An additive property of harmonic expectation was derived in the case of discrete random variable from the classical definition of harmonic expectation. Here, the same has been derived from the additive property of arithmetic expectation. This derivation of the additive property of harmonic expectation, along with numerical example, has been presented in this article.

Keywords: Discrete Random Variable, Harmonic expectation, Additive Property, Second Derivation.

1. INTRODUCTION

In statistics expectation is a theoretical concept associated with a variable random in nature and its measure or equivalently its definition [1 , 3 , 7 , 15 , 17 – 20 , 23] is based on the measure/definition of the concept of average [2 , 8]. Three definitions of expectation, developed based on the three classical means due to Pythagoras [6 , 8 , 10 , 13] namely arithmetic mean [4 , 5 , 24], geometric mean [4 , 5 , 21] & harmonic mean [4 , 5 , 16], are respectively arithmetic expectation, geometric expectation & harmonic expectation [9].

Arithmetic expectation, which had been developed first and whose several properties were identified, is also termed as mathematical expectation or simply expectation in standard literature of statistics [1 , 13 – 15 , 18 , 19 , 23]. Several similar properties of geometric expectation & harmonic expectation are yet to be identified. In the meantime, one property of geometric expectation has been identified which is its multiplicative property [11]. In another study, one property of harmonic expectation has been identified which is its additive property [12]. The additive property derived was derived in the case of discrete random variable from the classical definition of harmonic expectation. Here, the same has been derived from the additive property of arithmetic expectation. This derivation of the additive property of harmonic expectation, along with numerical example, has been presented in this article.

2. ARITHMETIC AND HARMONIC EXPECTATIONS

Definition

If X is a positive random variable assuming the values

$$x_1 , x_2 , \dots \dots \dots .. , x_M ,$$

with respective probabilities

$$p_1 , p_2 , \dots \dots \dots .. , p_M$$

i.e. $P(X = x_i) = p_i$,

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then EA(X), the arithmetic expectation of X, is defined by

$$\mathrm{EA}(X) = \sum_{i=1}^{M} p_i\, x_i$$

while the harmonic expectation of X, denoted by EH(X), is defined by

$$\mathrm{EH}(X) = \frac{1}{\sum_{i=1}^{M} p_i \frac{1}{x_i}}$$

Definitions of EH(X) & EA(X) imply that

$$E_H(X) = \frac{1}{E_A(\frac{1}{X})}$$

which implies,

$$\frac{1}{E_H(\frac{1}{X})} = E_A(X)$$

This means, the harmonic expectation of X is the reciprocal of the arithmetic expectation of the reciprocal of X and vice versa.

3. HARMONIC EXPECTATION-ADDITIVE PROPERTY

Additive property of harmonic expectation can be stated as follows:

If

$$X_1, X_2, \ldots\ldots\ldots, X_k$$

are k discrete random variables such that all assume non-zero values then

$$\frac{1}{E_H\left(\frac{1}{X_1+X_2+\ldots\ldots+X_k}\right)} = \frac{1}{E_H(\frac{1}{X_1})} + \frac{1}{E_H(\frac{1}{X_2})} + \ldots\ldots + \frac{1}{E_H(\frac{1}{X_k})}$$

Derivation

If X is a discrete random variable assuming non-zero values then the arithmetic expectation of X is the reciprocal of the harmonic expectation of the reciprocal of X (i.e. of $\frac{1}{X}$) i.e.

$$\frac{1}{E_H(\frac{1}{X})} = E_A(X)$$

Similarly, if Y is another discrete random variable assuming non-zero values the arithmetic expectation of Y is the reciprocal of the harmonic expectation of the reciprocal of Y (i.e. of $\frac{1}{Y}$) i.e.

$$\frac{1}{E_H(\frac{1}{Y})} = E_A(Y).$$

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Accordingly, the arithmetic expectation of $(X+Y)$ is the reciprocal of the harmonic expectation of the reciprocal of $(X+Y)$ (i.e. of $\frac{1}{X+Y}$) i.e.

$$E_A(X+Y) = \frac{1}{E_H(\frac{1}{X+Y})}$$

But by the additive property of arithmetic expectation, the arithmetic expectation of $(X+Y)$ is the sum of the individual arithmetic expectations of X & Y i.e.

$$E_A(X+Y) = E_A(X) + E_A(Y)$$

All these together imply that

$$\frac{1}{E_H(\frac{1}{X+Y})} = \frac{1}{E_H(\frac{1}{X})} + \frac{1}{E_H(\frac{1}{Y})}$$

Now, suppose that

$$X_1 , X_2 , \ldots\ldots\ldots , X_k$$

are k discrete random variables such each of them assume non-zero values then by the same logic as in the case of the variable X,

$$E_A(X_1) = \frac{1}{E_H(\frac{1}{X_1})}, \; E_A(X_2) = \frac{1}{E_H(\frac{1}{X_2})}, \; \ldots\ldots\ldots, E_A(X_k) = \frac{1}{E_H(\frac{1}{X_k})}$$

But by the generalized additive property of arithmetic expectation, the arithmetic expectation of the sum of $X_1 , X_2 , \ldots\ldots , X_k$ is the sum of their individual arithmetic expectations i.e.

$$E_A(X_1 + X_2 + \ldots.. + X_k) = E_A(X_1) + E_A(+X_2) + \ldots\ldots. + E_A(X_k)$$

Therefore,

$$\frac{1}{E_H(\frac{1}{X_1+X_2+\ldots\ldots..+X_k})} = \frac{1}{E_H(\frac{1}{X_1})} + \frac{1}{E_H(\frac{1}{X_2})} + \ldots\ldots.. + \frac{1}{E_H(\frac{1}{X_k})}$$

4. NUMERICAL EXAMPLE

Let us consider the random experiment of single throwing of an unbiased dice and define 3 random variables by

X = even integer , Y = odd integer & Z = integer whose square root is an integer

Here X assume the 3 values

2 , 4 , 6

with

$$P(X = 2) = 1/3\ ,\ P(X = 4) = 1/3,\ P(X = 6) = 1/3\,.$$

Y assume the 3 values

$$1\,,3\,,5$$

with

$$P(Y = 1) = 1/3\ ,\ P(Y = 3) = 1/3\ ,\ P(X = 5) = 1/3\,.$$

Z assume the 2 values

$$1\,,4$$

with

$$P(Z = 1) = ½\ ,\ P(Z = 4) = ½\,.$$

The variable the variable X + Y assumes the values

$$3\,,5\,,7\,,9\,,11$$

with

$$P(X + Y = 3) = 1/9\ ,\ P(X + Y = 5) = 2/9,\ P(X + Y = 7) = 3/9,$$
$$P(X + Y = 9) = 2/9\ ,\ P(X + Y = 11) = 1/9\,.$$

The variable the variable Y + Z assumes the values

$$2\,,4\,,5\,,6\,,7\,,9$$

with

$$P(Y + Z = 2) = 1/6\ ,\ P(Y + Z = 4) = 1/6,\ P(Y + Z = 5) = 1/6,$$
$$P(Y + Z = 6) = 1/6\ ,\ P(Y + Z = 7) = 1/6,\ P(Y + Z = 7) = 1/6\ .$$

The variable the variable X + Z assumes the values.

$$3\,,5\,,6\,,7\,,8\,,10$$

with

$$P(X + Z = 3) = 1/6\ ,\ P(X + Z = 5) = 1/6,\ P(X + Z = 6) = 1/6,$$
$$P(X + Z = 7) = 1/6\ ,\ P(X + Z = 8) = 1/6,\ P(X + Z = 10) = 1/6\ .$$

The variable the variable X + Y + Z assumes the values.

$$4\,,6\,,7\,,8\,,9\,,10\,,11\,,12\,,13\,,15$$

with

$$P(X + Y + Z = 4) = 1/10\ ,\ P(X + Y + Z = 6) = 1/10\ ,\ P(X + Y + Z = 7) = 1/10\ ,$$
$$P(X + Y + Z = 8) = 1/10\ ,\ P(X + Y + Z = 9) = 1/10\ ,\ P(X + Y + Z = 10) = 1/10\ ,$$
$$P(X + Y + Z = 11) = 1/10\ ,\ P(X + Y + Z = 12) = 1/10\ ,\ P(X + Y + Z = 13) = 1/10\ ,$$
$$P(X + Y + Z = 15) = 1/10\,.$$

From computations, it is found that

$$E_H(\frac{1}{X}) = 0.25, \quad \frac{1}{E_H(\frac{1}{X})} = 4.0,$$

$$E_H(\frac{1}{Y}) = 0.3333333333333, \quad \frac{1}{E_H(\frac{1}{Y})} = 3.0,$$

$$E_H(\frac{1}{Z}) = 0.4, \quad \frac{1}{E_H(\frac{1}{Z})} = 2.5$$

$$E_H(\frac{1}{X+Y}) = 0.1428571428571428571428571428571\,4, \quad \frac{1}{E_H(\frac{1}{X+Y})} = 7.0,$$

$$E_H(\frac{1}{Y+Z}) = 0.18181818181818181818181818181818, \quad \frac{1}{E_H(\frac{1}{Y+Z})} = 5.5,$$

$$E_H(\frac{1}{X+Z}) = 0.1538461538461538461538461538461\,5, \quad \frac{1}{E_H(\frac{1}{X+Z})} = 6.5,$$

$$E_H(\frac{1}{X+Y+Z}) = 0.10526315789473684210526315789474, \quad \frac{1}{E_H(\frac{1}{X+Y+Z})} = 9.5.$$

Note that

$$\frac{1}{E_H(\frac{1}{X})} + \frac{1}{E_H(\frac{1}{Y})} = 4.0 + 3.0 = 7.0 = \frac{1}{E_H(\frac{1}{X+Y})},$$

$$\frac{1}{E_H(\frac{1}{Y})} + \frac{1}{E_H(\frac{1}{Z})} = 3.0 + 2.5 = 5.5 = \frac{1}{E_H(\frac{1}{Y+Z})},$$

$$\frac{1}{E_H(\frac{1}{X})} + \frac{1}{E_H(\frac{1}{Z})} = 4.0 + 2.5 = 6.5 = \frac{1}{E_H(\frac{1}{X+Z})},$$

$$\frac{1}{E_H(\frac{1}{X})} + \frac{1}{E_H(\frac{1}{Y})} + \frac{1}{E_H(\frac{1}{Z})} = 4.0 + 3.0 + 2.5 = 9.5 = \frac{1}{E_H(\frac{1}{X+Y+Z})} .$$

5. CONCLUSION

The additive property of harmonic expectation was derived from its classical in earlier study [12]. In this study, it has been derived from the additive property of arithmetic expectation. The aim of this study was to verify whether the two tracks of derivation yield the same result and in the study it has been found so. Thus, we can be more confident on the validity of the result on this property of harmonic expectation.

In this connection, it is to be mentioned that each of arithmetic mean, geometric mean & harmonic mean can be defined in terms of another one among the three means. Consequently, it is possible to define each arithmetic expectation, geometric expectation & harmonic expectation in terms of another one among the three expectations. This leads to thinking of whether it is possible to derive a property of each of them from some property of another one among them.

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