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Rhythmic Additive Property of Harmonic Expectation

Dhritikesh Chakrabarty

Independent Researcher (Ex Associate Professor of Statistics, Handique Girls' College) Guwahati, Assam, India.

Abstract - Recently, a study has been done on searching for multiplicative property of geometric expectation. Here, attempt has been made on searching for additive property of harmonic expectation. The output obtained in the attempt has been presented in this article.

Keywords: Harmonic expectation, Discrete Random Variable, Additive Property.

1. INTRODUCTION

Expected value or expectation, a concept associated to random variable, had originally been defined as the the weighted arithmetic mean of its all possible values with their respective probabilities as the corresponding weights and termed as mathematical expectation and/or simply expectation [1 – 3 , 8 , 10 – 12]. It was termed as arithmetic expectation since it is based on arithmetic mean [4 , 6 , 15] one among the three Pythagorean means [4 , 6]. On the other hand, the concept of expectation was introduced and defined on the basis of the other two Pythagorean means namely geometric mean [4 , 6 ,13] and harmonic mean also [4 , 6]. The two concepts introduced on the basis of these two means were termed as geometric expectation and harmonic expectation respectively [5]. Some important and useful properties of arithmetic expectation (which is simply termed as "mathematical expectation" in the existing literature of statistics) had already been derived [8 , 9 , 11 , 14 , 16]. Recently, a study has been done on searching for multiplicative property of geometric expectation [7]. Here, attempt has been made on searching for additive property of harmonic expectation. The output obtained in the attempt has been presented in this article.

2. ADDITIVE PROPERTY OF HARMONIC EXPECTATION

Table -1: The following abbreviations/notations have been used in this article:

Abbreviation/Notation	Indication/Descri
drv	Discrete Random Variable
AE	Arithmetic Expectation
GE	Geometric Expectation
HE	Harmonic Expectation
$E_H(X)$	Harmonic Expectation of X

If a drv X assumes the values

$x_1 , x_2 , \dots \dots \dots .. , x_M ,$

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none of which is zero so that harmonic expectation exists, with respective probabilities

$$p_1, p_2, \dots \dots \dots, p_M$$

i.e. $P(X = x_i) = p_i$,

then EH(X) is defined by

$$E_H(X) = \frac{1}{\sum_{i=1}^{M} p_i . \frac{1}{x_i}}$$

[5], which implies,

$$\frac{1}{E_H(\frac{1}{X})} = \sum_{i=1}^{M} p_i . x_i$$

Similarly, the harmonic expectation of another drv Y which assumes the non-zero values

$y_1, y_2, \dots \dots \dots, y_N$

with respective probabilities

$q_1, q_2, \dots \dots \dots, q_N$

i.e. $P(Y = y_1) = q_j$,

is defined by

$$E_H(Y) = \frac{1}{\sum_{i=1}^{N} q_j . \frac{1}{y_j}}$$

which implies,

$$\frac{1}{E_H(\frac{1}{Y})} = \sum_{j=1}^{N} q_j . y_j$$

Now, let us consider the variable

$$\frac{1}{X + Y}$$

If

$P(X = x_i, Y = y_j) = r_{ij}$

then

$$\frac{1}{E_H(\frac{1}{X+Y})} = \sum_{i=1}^{M} \sum_{j=1}^{N} r_{ij}(x_i + y_j)$$

But

$$\sum_{j=1}^{N} r_{ij} = \sum_{j=1}^{N} P(X = x_i , Y = y_j) = P(X = x_i) = p_i$$

& $$\sum_{i=1}^{M} r_{ij} = \sum_{i=1}^{M} P(X = x_i , Y = y_j) = P(Y = y_j) = q_j$$

Therefore,

$$\frac{1}{E_H\left(\frac{1}{X+Y}\right)} = \sum_{i=1}^{M}\left(\sum_{j=1}^{N} r_{ij}\right)x_i + \sum_{j=1}^{N}\left(\sum_{i=1}^{M} r_{ij}\right)y_j$$

which implies,

$$\frac{1}{E_H\left(\frac{1}{X+Y}\right)} = \sum_{i=1}^{M} p_i x_i + \sum_{j=1}^{N} q_j y_j$$

Hence,

$$\frac{1}{E_H\left(\frac{1}{X+Y}\right)} = \frac{1}{E_H\left(\frac{1}{X}\right)} + \frac{1}{E_H\left(\frac{1}{Y}\right)}$$

Now, if X, Y & Z are three discrete random variables such that all assume non-zero values then

$$\frac{1}{E_H\left(\frac{1}{X+Y+Z}\right)} = \frac{1}{E_H\left\{\frac{1}{X+(Y+Z)}\right\}} = \frac{1}{E_H\left(\frac{1}{X}\right)} + \frac{1}{E_H\left(\frac{1}{Y+Z}\right)}$$

Hence,

$$\frac{1}{E_H\left(\frac{1}{X+Y+Z}\right)} = \frac{1}{E_H(X)} + \frac{1}{E_H\left(\frac{1}{Y}\right)} + \frac{1}{E_H\left(\frac{1}{Z}\right)}$$

By the application the mathematical induction, one can obtain that if

$X_1 , X , \ldots\ldots\ldots , X_n$

are n discrete random variables such that all assume non-zero values then

$$\frac{1}{E_H\left(\frac{1}{X_1+X_2+\ldots\ldots+X_n}\right)} = \frac{1}{E_H\left(\frac{1}{X_1}\right)} + \frac{1}{E_H\left(\frac{1}{X_2}\right)} + \ldots\ldots\ldots + \frac{1}{E_H\left(\frac{1}{X_n}\right)}$$

Thus, the following theorem, interpretable as additive property of harmonic expectation, has been obtained:

Theorem (2.1):

If

$X_1 , X , \ldots\ldots\ldots , X_n$

are n discrete random variables such that all assume non-zero values then

$$\frac{1}{E_H(\frac{1}{X_1+X_2+.......+X_n})} = \frac{1}{E_H(\frac{1}{X_1})} + \frac{1}{E_H(\frac{1}{X_2})} + ...+ \frac{1}{E_H(\frac{1}{X_n})}$$

3. NUMERICAL EXAMPLE

Suppose that an unbiased dice is thrown once.

If X is a drv denoting the number which is the half of the number occurred in the throw then the probabilities of the values assumed by it are

P(X = 0.5) = 1/6 , P(X = 1.0) = 1/6, P(X = 1.5) = 1/6 ,

P(X = 2.0) = 1/6 , P(X = 2.5) = 1/6 , P(X = 3.0) = 1/6 .

In this case, it is found from calculation that

$$E_H(\frac{1}{X}) = 0.5714285714285714285714285714285714285 7 ,$$

$$\frac{1}{E_H(\frac{1}{X})} = 1.75$$

Similarly, if Y is another random variable denoting the number which is the quarter of the number occurred in the throw then the probabilities of the values assumed by it are

P(Y = 0.25) = 1/6 , P(Y = 0.5) = 1/6 , P(Y = 0.75) = 1/6 ,

P(Y = 1.0) = 1/6 , P(Y = 1.25) = 1/6 , P(Y = 1.5) = 1/6 .

are all equal and is 1/6.

In this case, it is found from calculation that

$$E_H(\frac{1}{Y}) = 1.142857142857142857142857142857 1 ,$$

$$\frac{1}{E_H(\frac{1}{Y})} = 0.875$$

Now, the probabilities of the values assumed by $\frac{1}{X+Y}$ are as follows:

P($\frac{1}{X+Y}$ = 0.75) = P($\frac{1}{X+Y}$ = 1.0) = P($\frac{1}{X+Y}$ = 4.25) = P($\frac{1}{X+Y}$ = 4.5) = 1/36 ,

P($\frac{1}{X+Y}$ = 1.25) = P($\frac{1}{X+Y}$ = 1.5) = P($\frac{1}{X+Y}$ = 3.75) = P($\frac{1}{X+Y}$ = 4.0) = 1/18 ,

P($\frac{1}{X+Y}$ = 1.75) = P($\frac{1}{X+Y}$ = 2.0) = P($\frac{1}{X+Y}$ = 2.25) = P($\frac{1}{X+Y}$ = 2.5) = 1/12 ,

P($\frac{1}{X+Y}$ = 2.75) = P($\frac{1}{X+Y}$ = 3.0) = P($\frac{1}{X+Y}$ = 3.25) = P($\frac{1}{X+Y}$ = 3.5) = 1/12 .

In this case, it is found from calculation that

$$E_H\left(\frac{1}{X+Y}\right) = 0.380952380952380952380952380952380952 38$$

,

$$\frac{1}{E_H\left(\frac{1}{X+Y}\right)} = 2.625$$

Note that

$$\frac{1}{E_H\left(\frac{1}{X}\right)} + \frac{1}{E_H\left(\frac{1}{Y}\right)} = 1.75 + 0.875 = 2.625 = \frac{1}{E_H\left(\frac{1}{X+Y}\right)}$$

4. CONCLUSION

The additive property of arithmetic expectation, termed as **Theorem (2.1** expectation or simply expectation in standard literature in statistics, [8 , 9 , 11 , 14 , 16] can be summarized as

$\boldsymbol{AE}$ of Sum of $\boldsymbol{Variable}$ s = Sum of $\boldsymbol{AE}$ of $\boldsymbol{Variable}$ s

Theorem (2.1), describing the additive property of harmonic expectation, can be summarized as

$\frac{1}{\boldsymbol{HE} \text{ of } \left(\frac{1}{\text{Sum of } \boldsymbol{Variable} \text{ s}}\right)}$ = Sum of $\left\{\frac{1}{\boldsymbol{HE} \text{ of } \left(\frac{1}{\boldsymbol{Variable}}\right)}\right\}$s

This is the rhythm lying in the additive property of harmonic expectation. For this reason, additive property of harmonic expectation has been termed here as the **Rhythmic Additive Property of Harmonic Expectation**.

It is to be mentioned that the additive property of harmonic expectation has here been derived in the case of a discrete random variable. Derivation of this property is to be done in the case of continuous random variable.

Like arithmetic expectation, there may be more properties of harmonic expectation. Study can be made on searching for these properties if exist.

REFERENCES

[1] A N Shiryaev, V S Korolyuk, S Watanabe & M Fukushima (1992): "Probability Theory and Mathematical Statistics", Pages: 456. https://doi.org/10.1142/1780.

[2] Blume Lawrence, & James Jordan (1986): "Introduction to Expectations Equilibrium", Lecture Notes in Economics and Mathematical Systems, 206 –12. Berlin. http://dx.doi.org/10.1007/978-3-642-51645-0_14.

[3] Dazhi Zhang, Ouyang He, E. Stanley Lee, & Ronald R. Yager (1993): "On Fuzzy Random Sets and Their Mathematical Expectations", Information Sciences, 72(1-2), 123 – 42. ttp://dx.doi.org/10.1016/0020-0255(93)90032-h.

[4] Dhritikesh Chakrabarty (2016): “Pythagorean Mean: Concept behind the Averages and Lot of Measures of Characteristics of Data”, NaSAEAST-2016, Abstract ID: CMAST_NaSAEAST (Inv)-1601). DOI: 10.13140/RG.2.2.27022.57920 .

[5] Dhritikesh Chakrabarty (2024): “Idea of Arithmetic, Geometric and Harmonic Expectations”, Partners Universal International Innovation Journal (PUIIJ), 02(01), 119 – 124. www.puiij.com . DOI:10.5281/zenodo.10680751. https://www.researchgate.net/publication/378658532_Idea_of_Arithmetic_Geometric_and_Harmonic_Expectations .

[6] Dhritikesh Chakrabarty (2024): “Extended Inequality Satisfied by Pythagorean Classical means”, Partners Universal International Innovation Journal (PUIIJ), (ISSN: 2583-9675), 02(04), 15 – 18. www.puiij.com . DOI: 10.5281/zenodo.13621318 .

[7] Dhritikesh Chakrabarty (2024): “Beautiful Multiplicative Property of Geometric Expectation”, Partners Universal International Innovation Journal (PUIIJ), (ISSN: 2583-9675), 02(02), 92 – 98. www.puiij.com . DOI: 10.5281/zenodo.10999414 .

[8] Epps T. W. (2013): “Mathematical Expectation”, Probability and Statistical Theory for Applied Researchers, 127 – 234. https://doi.org/10.1142/9789814513166_0003.

[9] Hasan O., Tahar S. (2007): “Verification of Expectation Properties for Discrete Random Variables in HOL”, In: Schneider, K., Brandt, J. (eds) Theorem Proving in Higher Order Logics. TPHOLs 2007. Lecture Notes in Computer Science, 4732, Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-74591-4_10 .

[10] Kay S. M. (2012): “Expected Values for Discrete Random Variables”, In: Intuitive Probability and Random Processes Using MATLAB, Springer, Boston, MA. https://doi.org/10.1007/0-387-24158-2_6 .

[11] Ramalingam Shanmugam, Rajan Chattamvelli (2015): “Mathematical Expectation”, Wiley Online Library, Chapter 8, https://doi.org/10.1002/9781119047063.ch8 .

[12] Schmeidler D., Wakker P. (1990): “Expected Utility and Mathematical Expectation”, In: Eatwell, J., Milgate, M., Newman, P. (eds) Utility and Probability, The New Palgrave. Palgrave Macmillan, London. https://doi.org/10.1007/978-1-349-20568-4_10 .

[13] Stević S. (2011): “Geometric Mean”, In: Lovric, M. (eds) International Encyclopedia of Statistical Science, Springer, Berlin, Heidelberg, https://doi.org/10.1007/978-3-642-04898-2_644 , pp 608 – 609.

[14] Wei Yongqing & Peiyu Liu (2006): “Properties of Minimal Mathematical Expectations”, Applied Mathematics Letters 19, No. 1, 15 – 21. http://dx.doi.org/10.1016/j.

[15] Yadolah Dodge (2008): “Arithmetic Mean”, In: The Concise Encyclopedia of Statistics, Springer, New York, NY, https://doi.org/10.1007/978-0-387-32833-1_12, pp 15 – 18.

[16] Zun Shan (2016): “Probability and Expectation”, “Probability and Expectation”, Mathematical Olympiad And Competitions”, Mathematical Olympiad Series Book 14), ISBN-978-981314148. https://doi.org/10.1142/10074 .