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Extended Inequality Satisfied by Pythagorean Classical Means

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Abstract - The great mathematician Pythagoras who is the inventor of the three basic measures of average namely Arithmetic Mean, Geometric Mean & Harmonic Mean, which are termed as the Pythagorean classical means, derived an inequality satisfied by these three measures in a domain of numbers. An inequality has here been derived which is satisfied by the three Pythagorean classical means in a wider domain of numbers. This article presents the derivation of this extended inequality with numerical example.

Keywords: Average, Pythagorean Classical Means, Extended Domain of Numbers, Inequality.

1. INTRODUCTION

The concept of average [1 , 3 , 17] was introduced as a means of describing a characteristic of a class of individuals overall. Average is calculated by its measure from a list/set of numbers (or equivalently data in statistical literature).

The great mathematician Pythagoras [2 , 3 , 11] had invented the three basic measures of average namely Arithmetic Mean (AM), Geometric Mean (GM) & Harmonic Mean (HM) [3 , 6 , 7 , 8 , 10 , 12 , 14 , 15 , 18] These were later termed as Pythagorean classical means [10]. Each of these three measures of average was derived on the basis of some philosophy [4 , 7].

In statistics, these the three Pythagorean means are used as the basis of measures of central tendency of data [9 , 16 , 19 , 20].

Moreover, these three are used and/or can be used as the tool of constructing measures of various characteristics of data [3].

Pythagoras derived an inequality satisfied by the three Pythagorean classical means in a domain of numbers [4 , 8 , 13].

However, numerical data are not always limited to this domain of numbers within which the inequality derived by Pythagoras is valid. Thus, there is necessity of searching for an inequality to be satisfied by these three classical means in a wider domain of numbers and/or to extend/modify the inequality derived by Pythagoras for in a wider domain of numbers. The great mathematician Pythagoras who is the inventor of the three basic measures of average namely Arithmetic Mean, Geometric Mean & Harmonic Mean, which are termed as the classical Pythagorean means derived an inequality satisfied by these three measures in a domain of numbers. An inequality has here been derived which is satisfied by the three Pythagorean means in a wider domain of numbers. This inequality can be interpreted as an extension of the inequality established by Pythagoras. This article presents the derivation of this extended inequality with numerical example. An inequality has here been derived which is satisfied by the three Pythagorean classical means in a wider domain of numbers. This inequality can be interpreted as an extension of the inequality established by Pythagoras. This article presents the derivation of this extended inequality with numerical example. I

2. PYTHAGOREAN CLASSICAL MEANS

The three Pythagorean classical means are Arithmetic Mean (AM), Geometric Mean (GM) & Harmonic

Let

$x_1 , x_2 , \ldots\ldots , x_n$

be a list/set of n real numbers..

Arithmetic Mean of the list of numbers is a number A such that

$$x_1 + x_2 + \ldots\ldots\ldots + x_n = A + A + \ldots\ldots\ldots + A \qquad (2.1)$$

i.e.

$$A = \frac{1}{n} (x_1 + x_2 + \ldots\ldots\ldots + x_n) \qquad (2.2)$$

Geometric Mean of the list of numbers is a number G such that

$$x_1 . x_2 . \ldots\ldots . x_n = G . G \ldots\ldots\ldots G \qquad (2.3)$$

i.e.

$$G = (x_1 . x_2 . \ldots\ldots . x_n)1/n \qquad (2.4)$$

provided $x_1 , x_2 , \ldots\ldots , x_n$ are all positive.

Harmonic Mean of the list of numbers is a number H such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots\ldots + \frac{1}{x_n} = \frac{1}{H} + \frac{1}{H} + \ldots\ldots\ldots + \frac{1}{H} \qquad (2.5)$$

i.e.

$$H = \frac{1}{\frac{1}{n}(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots\ldots + \frac{1}{x_n})} \qquad (2.6)$$

provided $x_1 , x_2 , \ldots\ldots , x_n$ are all non-zero.

Now, let us use the notations

AM $(x_1 , x_2 , \ldots\ldots , x_n)$, GM $(x_1 , x_2 , \ldots\ldots , x_n)$ & HM $(x_1 , x_2 , \ldots\ldots , x_n)$

to denote the Arithmetic Mean, Geometric Mean & Harmonic Mean respectively of

$x_1 , x_2 , \ldots\ldots , x_n$

Note (2.1):

Min $(x_1 , x_2 , \ldots\ldots , x_n)$ < AM $(x_1 , x_2 , \ldots\ldots , x_n)$ < Max $(x_1 , x_2 , \ldots\ldots , x_n)$,

Min $(x_1 , x_2 , \ldots\ldots , x_n)$ < GM $(x_1 , x_2 , \ldots\ldots , x_n)$ < Max $(x_1 , x_2 , \ldots\ldots , x_n)$

& Min $(x_1 , x_2 , \ldots\ldots , x_n)$ < HM $(x_1 , x_2 , \ldots\ldots , x_n)$ < Max $(x_1 , x_2 , \ldots\ldots , x_n)$

i.e. each of

AM $(x_1 , x_2 , \ldots\ldots , x_n)$, GM $(x_1 , x_2 , \ldots\ldots , x_n)$ & HM $(x_1 , x_2 , \ldots\ldots , x_n)$

lies within the smallest of $x_1, x_2, \ldots\ldots, x_n$ & and the largest of $x_1, x_2, \ldots\ldots, x_n$ provided $x_1, x_2, \ldots\ldots, x_n$ are not identical.

2.1. Pythagorean Inequality

Pythagorean established that

AM $(x_1, x_2, \ldots\ldots, x_n)$ > GM $(x_1, x_2, \ldots\ldots, x_n)$ > HM $(x_1, x_2, \ldots\ldots, x_n)$

when $x_1, x_2, \ldots\ldots, x_n$ are positive, real and not all identical.

In particular, when $x_1, x_2, \ldots\ldots, x_n$ are all identical then

AM $(x_1, x_2, \ldots\ldots, x_n)$ = GM $(x_1, x_2, \ldots\ldots, x_n)$ = HM $(x_1, x_2, \ldots\ldots, x_n)$

3. EXTENSION OF PYTHAGOREAN INEQUALITY

Suppose,

$x_1, x_2, \ldots\ldots, x_n$

are all positive and are not all identical.

Then

$-x_1, -x_2, \ldots\ldots, -x_n$

are all negative and are not all identical.

Now,

$$\text{AM}(-x_1, -x_2, \ldots\ldots, -x_n) = \frac{1}{n}\{(-x_1) + (-x_2) + \ldots\ldots\ldots + (-x_n)\}$$
$$= -\frac{1}{n}\{x_1 + x_2 + \ldots\ldots\ldots + x_n\}$$

i.e. $\text{AM}(-x_1, -x_2, \ldots\ldots, -x_n) = -\text{AM}(x_1, x_2, \ldots\ldots, x_n)$ (3.1)

Similarly,

$$\text{HM}(-x_1, -x_2, \ldots\ldots, -x_n) = \frac{1}{\frac{1}{n}\{\frac{1}{(-x_1)} + \frac{1}{(-x_2)} + \ldots\ldots\ldots + \frac{1}{(-x_n)}\}}$$
$$= -\frac{1}{\frac{1}{n}(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots\ldots + \frac{1}{x_n})}$$

i.e. $\text{HM}(-x_1, -x_2, \ldots\ldots, -x_n) = -\text{HM}(x_1, x_2, \ldots\ldots, x_n)$ (3.2)

Let us now define geometric mean of the negative numbers

$-x_1, -x_2, \ldots\ldots, -x_n$

By the philosophy of mean as mentioned in Note (2.1),

GM $(-x_1, -x_2, \ldots\ldots, -x_n)$

is to lie lies within

the smallest of $-x_1, -x_2, \ldots\ldots, -x_n$ & and the largest of $-x_1, -x_2, \ldots\ldots, -x_n$.

Hence,

GM $(-x_1, -x_2, \ldots\ldots\ldots, -x_n)$

is negative.

By definition of geometric mean,

$$\text{GM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) = \{(-x_1).(-x_2).\ \ldots\ldots\ldots .(-x_n)\}1/n$$

$$= \{(-1).(-1).\ \ldots\ldots\ldots .(-1)\}1/n\ \{(x_1).(x_2).\ \ldots\ldots\ldots .(x_n)\}1/n$$

$$= \{(-1)\ n\}1/n\ \{x_1.x_2.\ \ldots\ldots\ldots .x_n\}1/n$$

i.e. $\text{GM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) = -\text{GM}\ (x_1.x_2.\ \ldots\ldots\ldots .x_n)$ (3.3)

Now,

$\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) > \text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

$\Rightarrow -\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) < -\text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

$\Rightarrow \text{AM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) < \text{GM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n)$

Similarly,

$\text{GM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) < \text{HM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n)$

Hence,

$\text{AM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) < \text{GM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n) < \text{HM}\ (-x_1, -x_2, \ldots\ldots\ldots, -x_n)$

i.e. $\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) < \text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) < \text{HM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

when $x_1, x_2, \ldots\ldots\ldots, x_n$ are all negative real numbers and are not all identical.

Thus, the relationship satisfied by AM, GM & AH can be stated in the form of a theorem as follows:

Theorem (2.1):

Arithmetic Mean, Geometric Mean and Harmonic Mean satisfy the relation

$\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) > \text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) > \text{HM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

when $x_1, x_2, \ldots\ldots\ldots, x_n$ are positive, real and not all identical,

$\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) < \text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) < \text{HM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

when $x_1, x_2, \ldots\ldots\ldots, x_n$ are negative, real and not all identical

and

$\text{AM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) = \text{GM}\ (x_1, x_2, \ldots\ldots\ldots, x_n) = \text{HM}\ (x_1, x_2, \ldots\ldots\ldots, x_n)$

when $x_1, x_2, \ldots\ldots\ldots, x_n$ are all non-zero real and identical.

4. NUMERICAL EXAMPLE

Let us take the numbers

1, 2, 3, 4 ,5 , 6 , 7 , 8 , 9 , 10

In this case

AM (1 , 2 , ….. , 9 , 10) = 5.5 ,

GM (1 , 2 , ….. , 9 , 10) = 4.528728688116764762203309 34

& HM (1 , 2 , ….. , 9 , 10) = 3.414171521474055006096734 86

i.e. AM (1 , 2 , ….. , 9 , 10) > GM (1 , 2 , ….. , 9 , 10) > HM (1 , 2 , ….. , 9 , 10)

Now, AM, GM & HM of

–1 , –2 , –3 , – 4 , –5 , – 6 , –7 , – 8 , – 9 , – 10

are as follows:

AM (–1 , –2 , ……. , – 9 , – 10) = –5.5 ,

GM (–1 , –2 , ……. , – 9 , – 10) = – 528728688116764762203309 34

& HM (–1 , –2 , ……. , – 9 , – 10) = –.414171521474055006096734 86

i.e. AM –1 , –2 , …. , – 9 , –10) > GM (–1 , –2 , …. , – 9 , –10) > HM (–1 , –2 , …. , – 9 , –10)

5. CONCLUSION

Following conclusions can be drawn from the above findings:

(1) Geometric mean exists and is defined for a list/set of numbers which are either all positive or all negative.

(2) Arithmetic mean, geometric mean and harmonic mean satisfy the inequality given in **Theorem (2.1)**.

(3) The inequality, described by **Theorem (2.1)**, can be interpreted as an extension of the inequality established by Pythagoras in the domain real numbers which are either all positive or all negative.

(4) Till this date, geometric mean has been treated as suitable for measuring central tendency of numerical data when all the numerical data are strictly positive. One significant findings of this study is that geometric mean can be also be suitable for measuring central tendency of numerical data when all the data are negative.

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