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Beautiful Multiplicative Property of Geometric Expectation

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Abstract - Concepts of geometric expectation and harmonic expectation of random variable were introduced in a recent study with formulating their definitions in the case of discrete random variable. It was thought that geometric expectation might carry some properties. A beautiful multiplicative property of geometric expectation has been derived in this study. The property has been derived in the case of discrete random variable. This article presents derivation of this property of geometric expectation with numerical example.

Keywords: Geometric expectation, Discrete Random Variable, Multiplicative Property.

1. INTRODUCTION

Expected value or expectation is a concept associated to random variable. A random variable assumes many possible values whose average is the expected value or expectation of the random variable [1 , 4 , 13 , 14 , 15 ,17]. Since each of the possible values corresponds to a probability, the corresponding probabilities can be regarded as and hence are to be used as the weights of the respected possible values in determining the expectation of the variable. Accordingly, expectation (also termed as mathematical expectation) of a random variable had been defined as the the weighted arithmetic mean of its all possible values with their respective probabilities as the corresponding weights [12 , 13 , 14 , 15 ,17].

Arithmetic mean is a measure of average of a list of numbers introduced by Pythagoras who had introduced two other measures of average namely geometric mean and harmonic mean [7]. Being two measures of average, geometric mean and harmonic mean can also be used as weights in defining expectation of a random variable. In a study, concepts of geometric expectation and harmonic expectation of random variable had been introduced with formulating their definitions with the help of geometric mean and harmonic mean respectively as weights [8 , 9 , 10]. Some important and useful properties of arithmetic expectation (which is simply termed as "mathematical expectation" in the existing literature of statistics) had already been derived [12 , 13 , 14 , 15 ,17]. It is assumed that geometric expectation carries some properties. It was thought that geometric expectation might carry some properties. A beautiful multiplicative property of geometric expectation has been derived in this study. The property has been derived in the case of discrete random variable. This article presents derivation of this property of geometric expectation with numerical example.

2. MULTIPLICATIVE PROPERTY OF GEOMETRIC EXPECTATION

Let X be a discrete random variable which assumes the M possible values

$x_1 , x_2 , \dots \dots \dots .. , x_M ,$

which are strictly positive so that geometric expectation exists, with respective probabilities

$$p_1 , p_2 , \dots \dots \dots , p_M$$

i.e. $P(X = x_1) = p_1$, $P(X = x_2) = p_2$, ... , $P(X = x_M) = p_M$.

Then, the geometric expectation [8 , 9 , 10] of X, denoted by EG(X), is defined by

$$EG(X) = \prod_{i=1}^{M} x_i^{\,p_i} \qquad (2.1)$$

Also, let Y be another discrete random variable which assumes the N possible positive values

$y_1 , y_2 , \dots \dots \dots , y_N$

with respective probabilities

$q_1 , q_2 , \dots \dots \dots , q_N$

i.e. $P(Y = y_1) = q_1$, $P(Y = y_2) = q_2$, ... , $P(Y = y_N) = q_N$.

Then, the geometric expectation of Y, denoted by EG(Y), is defined by

$$EG(Y) = \prod_{j=1}^{N} y_j^{\,q_j} \qquad (2.2)$$

Now let us consider XY, the product of the two random variable X &Y.

XY, being the product of random variables, is also a random variable.

XY assumes the MN possible values

$x_1y_1 , x_1y_2 , \dots\dots , x_1y_N$,

$x_2y_1 , x_2y_2 , \dots\dots , x_2y_N$,

....

$x_My_1 , x_My_2 , \dots\dots , x_My_N$.

If

$P(X = x_i , Y = y_j) = r_{ij}$

(i = 1 , 2 , ... , M & j = 1 , 2 , ... , N) ,

then geometric expectation of XY becomes

$$EG(XY) = \prod_{i=1}^{M} \prod_{j=1}^{N} (x_i y_j)^{r_{ij}}$$

But

$$\sum_{j=1}^{N} r_{ij} = \sum_{j=1}^{N} P(X = x_i , Y = y_j) = P(X = x_i) = p_i$$

(for i = 1 , 2 , ... , M)

& $$\sum_{i=1}^{M} r_{ij} = \sum_{i=1}^{M} P(X = x_i , Y = y_j) = P(Y = y_j) = q_j$$

(for j = 1 , 2 , ... , N)

Thus,

$$EG(XY) = \prod_{i=1}^{M} \prod_{j=1}^{N} (x_i y_j)^{r_{ij}}$$

$$= \{\prod_{i=1}^{M} \prod_{j=1}^{N} (x_i)^{r_{ij}}\}\{\prod_{j=1}^{N} \prod_{i=1}^{M} (y_j)^{r_{ij}}\}$$
$$= \{\prod_{i=1}^{M} (x_i)^{\sum_{j=1}^{N} r_{ij}}\}\{\prod_{j=1}^{N} (y_j)^{\sum_{i=1}^{M} r_{ij}}\}$$
$$= \{\prod_{i=1}^{M} x_i^{\,p_i}\}\{\prod_{j=1}^{N} y_j^{\,q_j}\}$$

Hence,

EG(XY) = EG(X) EG(Y) (2.3)

i.e. the geometric expectation of the product of two discrete random variables is the product of their individual geometric expectations.

Now, if X, Y & Z are three discrete random variables, then

G(XYZ) = EG{(XY)Z}

= EG(XY) EG(Z) , by equation (2.3)

Hence,

EG(XYZ) = EG(X) EG(Y) EG(Z) (2.4)

i.e. the geometric expectation of the product of three discrete random variables is the product of their individual geometric expectations.

By the application the mathematical induction, one can obtain that if

$X_1, X, \ldots\ldots, X_n$

are n discrete random variables, then

EG$(X_1X_2 \ldots\ldots X_n)$ = EG(X_1) EG(X_2) ……… EG(X_n) (2.5)

i.e. the geometric expectation of the product of a finite number of discrete random variables is the product of their individual geometric expectations.

Thus, the following multiplicative property of geometric expectation obtained which has been written in the form of theorem:

Multiplicative Theorem of Geometric Expectation

The geometric expectation of the product of a finite number of discrete random variables is the product of their individual geometric expectations i.e. if

$X_1, X, \ldots\ldots, X_n$

are n discrete random variables, then

EG$(X_1X_2 \ldots\ldots X_n)$ = EG(X_1) EG(X_2) ……… EG(X_n)

Note:

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It is not necessary that the random variables must be independent for the multiplicative /theorem/rule of geometric expectation to be held good. This theorem/rule holds good for random variables which are not independent also.

3. NUMERICAL EXAMPLE

Let a fair dice is thrown at random.

Then the possible 6 outcomes (the numbers to be appeared) are

1 , 1 , 3 , 4 , 5 , 6

which are mutually exclusive, equally likely and exhaustive.

Let us define random variables X which denotes the number which is 2 more than the number occurred in the throw.

Then the possible values assumed by X are

3 , 4 , 5 , 6 , 7 , 8 ,

which are mutually exclusive, equally likely and exhaustive.

The probability distribution of X obtained by the application of classical probability [2 – 6 , 11 , 16] is as follows:

Table -3.1 : (Probability Distribution of X)

X	3	4	5	6	7	8
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

In this case,

EG(X) = 31/6 × 41/6 × 51/6 × 61/6 × 71/6 × 81/6

= 5.2169309429791640182398012447017

Again let us define another random variables Y which denotes the number which is double of the number occurred in the throw.

Then the possible values assumed by Y are

2 , 4 , 6 , 8 , 10 , 12 ,

which are mutually exclusive, equally likely and exhaustive.

The probability distribution of Y obtained by the application of classical probability [2 – 6 , 11 , 16] is as follows:

Table -3.2 : (Probability Distribution of Y)

Y	2	4	6	8	10	12
P(Y)	1/6	1/6	1/6	1/6	1/6	1/6

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In this case,

EG(Y) = 21/6 × 41/6 × 61/6 × 81/6 × 101/6 × 121/6

= 5.98759033104781790982032113577789

Now, consider the product XY of X & Y.

XY, being the product of random variables, is also a random variable.

The possible values assumed by XY are

6 , 12 , 18 , 24 , 30 , 36 ,

8 , 16 , 24 , 32 , 40 , 48 ,

10 , 20 , 30 , 40 , 50 , 60 ,

12 , 24 , 36 , 48 , 60 , 72 ,

14 , 28 , 42 , 56 , 70 , 84 ,

16 , 32 , 48 , 64 , 80 , 96 ,

which are mutually exclusive, equally likely and exhaustive.

The probability distribution of XY obtained by the application of classical probability [2 – 6 , 11 , 16] is as follows:

Table -3.3 : (Probability Distribution of XY)

XY	6	8	10	12	14
P(XY)	1/36	1/36	1/36	1/18	1/36
XY	16	18	20	24	28
P(XY)	1/18	1/36	1/36	1/12	1/36
XY	30	32	36	40	42
P(XY)	1/18	1/18	1/18	1/18	1/36
XY	48	50	56	60	64
P(XY)	1/12	1/36	1/36	1/18	1/36
XY	70	72	80	84	96
P(XY)	1/36	1/36	1/36	1/36	1/36

In this case,

EG(XY) = 61/36 × 81/36 × 101/36 × 121/18 × 141/36 × 161/18 × 181/36 × 201/36 × 241/12 ×

281/36 × 301/18 × 321/18 × 361/18 × 401/18 × 421/36 × 481/12 × 501/36 × 561/36 ×

601/18 × 641/36 × 701/36 × 721/36 × 801/36 × 841/36 × 961/36

$= (6 \times 8 \times 10 \times 14 \times 18 \times 20 \times 28 \times 42 \times 50 \times 56 \times 64 \times 70 \times 72 \times 80 \times 84 \times 96)1/36 \times$

$(12 \times 16 \times 30 \times 32 \times 36 \times 40 \times 60)1/18 \times (24 \times 48)1/12$

$= 16576320815990046720000001/36 \times 159252480001/18 \times 11521/12$

$= 4.70720984691045146150779787145\,52 \times 3.6879291314139005285862283763377$

$\times 1.7993723388229389206002864381875$

$= 31.236845271926217543445962142483$

Note that

$EG(X)\ EG(Y) = 5.2169309429791640182398012447017 \times$

$5.9875903310478179098203211357789$

$= 31.236845271926217543445962142483$

$= EG(XY)$

4. CONCLUSION

It is an established property of arithmetic expectation that the arithmetic expectation of the product of two random variables is the product of their individual arithmetic expectations if and only if the two variables are independent. But the geometric expectation of the product of two random variables is the product of their individual geometric expectations irrespective of whether the variables are independent or not independent. This can be regarded as a merit of geometric expectation over arithmetic expectation.

Like arithmetic expectation, there may be more properties of geometric expectation which are required to be extracted in further research study.

Similarly, harmonic expectation is also to carry some properties. Study can be made on extracting these properties also.

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