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# Idea of Arithmetic, Geometric and Harmonic Expectations

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**Abstract -** The expected value of a random variable, which is the weighted average of its all possible values with their respective probabilities as the corresponding weights, had already been defined with the help of arithmetic mean and the definition was termed as mathematical expectation. The same has, in this study, been defined made with the help of geometric mean and harmonic mean. In order to be free from confusion, the existing definition of mathematical expectation will be termed as arithmetic expectation since it is based on arithmetic mean. The two definitions of mathematical expectation to be defined with the help of geometric mean and harmonic mean will be termed as geometric expectation and harmonic expectation respectively. This article describes these two definitions with numerical examples.

**Keywords:**Expectation, Arithmetic, Geometric, Harmonic.

## 1. INTRODUCTION

Any process (or phenomena or experiment), random in nature, knowing the outcome on any one realization of the process is not possible. Instead, one can think of what can be expected to happen. Accordingly, expected happening is based on the concept of probability (i.e. measure of possibility or measure of chance) and is described/explained by the concept of expected value (also called expectation, expectancy, and mathematical expectation).

The expected value of a random variable, which is the weighted average of its all possible values with their respective probabilities as the corresponding weights, had already been defined with the help of arithmetic mean and the definition was termed as mathematical expectation.

Probability, which is accepted as weight in defining expectation of a random variable, had been defined in six approaches namely Subjective Approach [2], Intuitive Approach [12 , 13], Classical Approach [3], Empirical/Statistical Approach [15 , 16] with extension to automatically or naturally happened experiments/phenomena [8 , 9], Axiomatic Approach [3 , 4 , 10 , 11] and Theoretical Approach [5].

As mentioned above, the existing definition of expected value of a random variable states that it is the weighted arithmetic mean of its all possible values with their respective probabilities as the corresponding weights. Arithmetic mean [7], introduced by Pythagoras, is a measure of average of a list of numbers. Pythagoras had introduced two other measures of average which are geometric mean and harmonic mean [7]. In fact, lot of measures of average has been developed so far [7]. Since it had been possible to define expectation of a random variable with the help of arithmetic mean which is a measure of average, it can also be possible to define the same with the help of other measures of average. That is why the same has here been defined on the basis of geometric mean and harmonic mean. In order to be free from confusion, the existing definition of mathematical expectation will be termed as arithmetic expectation since it is based on arithmetic mean. The two definitions of mathematical expectation to be defined with the help of geometric mean and harmonic mean will be termed as geometric expectation and harmonic expectation respectively. This article describes these two definitions with numerical examples.

### 1.1 Notation Used

- AM = Arithmetic Mean ,
- GM = Geometric Mean ,
- HM = Harmonic Mean ,
- AE = Arithmetic Expectation ,
- GE = Geometric Expectation ,
- HE = Harmonic Expectation ,
- AM (X) = Arithmetic Mean of X ,
- GM (X) = Geometric Mean of X ,
- HM (X) = Harmonic Mean of X ,
- $E_A$ (X) = Arithmetic Expectation of X ,
- $E_G$ (X) = Geometric Mean of X ,
- & $E_H$ (X) = Harmonic Mean of X ,

## 2. ARITHMETIC EXPECTATION

Let us consider a discrete finite random variable.

Suppose, X assumes the values

$$x_1 , x_2 , \dots \dots \dots .. , x_N$$

These N values constitute the population of the variable X.

AM (X), as introduced by Pythagoras [26 , 27 , 39], is defined by

$$AM(X) = AM\ (x_1 , x_2 , \dots \dots \dots .. , x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Now suppose, X assumes the values

$$x_1 , x_2 , \dots \dots \dots .. , x_N$$

with respective probabilities

$$p_1 , p_2 , \dots \dots \dots .. , p_N$$

AM (X), which in this case becomes the weighted arithmetic mean of X, is defined by

$$AM(X) = AM\ (x_1 , x_2 , \dots \dots \dots .. , x_N) = \frac{\sum_{i=1}^{N} p_i x_i}{\sum_{i=1}^{N} p_i}$$

Since $\sum_{i=1}^{N} p_i = 1$

therefore $AM(X) = AM\ (x_1 , x_2 , \dots \dots \dots .. , x_N) = \sum_{i=1}^{N} p_i x_i$

which is the mathematical expectation [53 , 59 , 60 , 61 , 69] of X, symbolically E(X).as already defined and hence

E(X) = $\sum_{i=1}^{n} x_i$ P(X $= x_i$) = $\sum_{i=1}^{n} p_i x_i$

Since this definition is based on population AM, it can be interpreted as arithmetic expectation of X.

Accordingly, arithmetic expectation of $X$ denoted by $E_A$(X) becomes

$E_A$(X) = $\sum_{i=1}^{n} x_i$ P(X $= x_i$) = $\sum_{i=1}^{n} p_i x_i$

**Geometric Expectation**:

GM (X), as introduced by Pythagoras [26 , 27 , 39], is defined by

$$GM(X) = GM\ (x_1 , x_2 , \dots \dots \dots ., \ x_N\ ) = (\prod_{i=1}^{N} x_i)^{1/N}$$

while the GM of X which assumes the values

$$x_1 , x_2 , \dots \dots \dots ., \ x_N$$

with respective probabilities

$$p_1 , p_2 , \dots \dots \dots ., \ p_N$$

is the weighted GM of X and is defined by

$$GM(X) = GM\ (x_1 , x_2 , \dots \dots \dots ., \ x_N\ ) = (\prod_{i=1}^{N} x_i^{\ p_i})^{\ \sum_{i=1}^{N} p_i}$$

i.e. $GM(X) = GM\ (x_1 , x_2 , \dots \dots \dots ., \ x_N\ ) = \prod_{i=1}^{N} x_i^{\ p_i}$ , since $\sum_{i=1}^{N} p_i = 1$

Since the population AM of X is its AE, the population GM of X can be regarded as its GE.

Hence, the GE of $X$ denoted by $E_G$(X) becomes

EG(X) = $\prod_{i=1}^{N} P(X = x_i)^{\ p_i} = \prod_{i=1}^{N} x_i^{\ p_i}$

Harmonic Expectation:

HM (X), as introduced by Pythagoras [26 , 27 , 39], is defined by

$$HM(X) = HM\ (x_1 , x_2 , \dots \dots \dots ., \ x_N) = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} \frac{1}{x_i}}$$

while the HM of X which assumes the values

$$x_1 , x_2 , \dots \dots \dots ., \ x_N$$

with respective probabilities

$$p_1 , p_2 , \dots \dots \dots ., \ p_N$$

is the weighted AH of X and is defined by

$$HM(X) = HM\ (x_1 , x_2 , \dots \dots ., \ x_N) = \frac{1}{\frac{1}{\sum_{i=1}^{N} p_i}\sum_{i=1}^{N} p_i \frac{1}{x_i}} = \frac{1}{\sum_{i=1}^{N} p_i \frac{1}{x_i}}$$

By the same logic as in the cases of AE and GE, the population HM of X can be regarded as its HE.

Hence, HE of $X$ denoted by $E_H(X)$ becomes

$$E_H(X) = \frac{1}{\sum_{i=1}^{N} P(X=x_i)\frac{1}{x_i}} = \frac{1}{\sum_{i=1}^{N} p_i\frac{1}{x_i}}$$

**Note:**

(1) $AM\ (x_1, x_2, \dots\dots\dots, x_N)$ is defined for any real values of $x_1, x_2, \dots\dots\dots, x_N$.

Accordingly, $E_A(X)$ is defined for any real valued random variable.

However, each of $GM\ (x_1, x_2, \dots\dots\dots, x_N)$ and $HM\ (x_1, x_2, \dots\dots\dots, x_N)$ is defined only when values of $x_1, x_2, \dots\dots\dots, x_N$.are strictly positive.

Hence, each of $E_G(X)$ and $E_H(X)$ is defined for strictly a positive valued random variable X.

(2) AM, GM and AH satisfy the inequality [26 , 27 , 39]

$$AM\ (x_1, x_2, \dots\dots, x_N) > GM\ (x_1, x_2, \dots\dots, x_N) > HM\ (x_1, x_2, \dots\dots, x_N)$$

i.e. AM(X) > GM(X) > HM(X)

Hence, $E_A(X) > E_G(X) > E_H(X)$

## 3. NUMERICAL EXAMPLE

The number of rainy days at a place in a month is a random variable which assumes several possible values (whether point values or interval values are taken into account) with corresponding probabilities of occurrences. Probability of occurrence of the same at New Delhi in the month June has been estimated [51] and found as follows:

**Table -3.1:**

| June | |
|---|---|
| Number/Interval of Rainy Days | Probability of occurrence |
| 1 | 0.0625 |
| 2 | 0.09385 |
| [3 , 5] | 0.5625 |
| [6 , 8] | 0.25 |
| 9 | 0.03125 |
| > 9 | 0 |

In order to find out the values of arithmetic expectation, geometric expectation and harmonic expectation rainy days, we are to construct the following table:

**Table -3.2:**

| Number/Interval of Rainy Days | Mid value ($x_i$) | Probability of occurrence ($p_i$) | Values of $p_i x_i$ | Values of $x_i^{p_i}$ | Values of $p_i \frac{1}{x_i}$ |
|---|---|---|---|---|---|
| 1 | 1 | 0.0625 | 0.0625 | 1 | 0.0625 |
| 2 | 2 | 0.09375 | 0.1875 | 1.0671404006768236181695211209928 | 0.046875 |
| 3 – 5 | 4 | 0.5625 | 2.25 | 2.1810154653305153184140213115214 | 0.140625 |
| 6 – 8 | 7 | 0.25 | 1.75 | 1.6265765616977857432112323454938 | 0.035714 |
| 9 | 9 | 0.03125 | 0.28125 | 1.0710754830729144691232476346997 | 0.003472 |

Here $\sum_{i=1}^{N} p_i x_i = 4.53125$ ,

$\prod_{i=1}^{N} x_i^{p_i} = 4.0548509573441987687123105790392$

& $\sum_{i=1}^{N} p_i \frac{1}{x_i} = 0.28918650793650793650793650793651$

Hence, $E_A(X) = \sum_{i=1}^{N} p_i x_i = 4.53125$ ,

$E_G(X) = \prod_{i=1}^{N} x_i^{p_i} = 4.0548509573441987687123105790392$

& $E_H(X) = \sum_{i=1}^{N} p_i \frac{1}{x_i} = 3.4579759862778730703259005145797$

Accordingly, the estimated values of arithmetic expectation, geometric expectation and harmonic expectation of number of rainy days in the month June at New Delhi are

4.53125 , 4.0548509573441987687123105790392 & 3.4579759862778730703259005145797

respectively.

## 4. CONCLUSION

The existing definition of expectation, as mentioned in the 2nd paragraph of Section 1, can be regarded as a special definition of expectation but not its general definition while expectation in general can be defined as an weighted average of all the possible values assumed by a random variable, weighted by the respective probabilities of the possible values of the variable.

Of course, there is no certainty that the expected value will appear in the data set of a sample drawn for some purpose and it is also not the value one can "expect" to get in reality.

In reality, data are not of the same type in every situation. Accordingly, arithmetic expectation may not be proper to apply in every type of data set. There are or there may be some situations where geometric expectation may be proper to be applied in the respective data set. Similar is the case for harmonic expectation also.

At this stage, it is to be mentioned that the three definitions of expectation discussed here may not suit all types of data set. Thus, there is necessity of developing more measures/definitions of expectation.

On the whole, it can right now be concluded that the extracted information on the two definitions of expectation namely geometric expectation and harmonic expectation in addition to the existing one namely arithmetic expectation can be regarded as an important and useful outcome of a fundamental research as implied by the meaning of fundamental research [6].

## REFERENCES

[1] Al. A. Tchouproff (1918): "On the Mathematical Expectation of the Moments of Frequency Distributions" Biometrika, 12(1/2), 140 – 69. https://doi.org/10.2307/2331933 . Accessed 10 Feb. 2024.
[2] Bernard G. A. (1958): "Thomas Bayes Essay towards Solving a Problem in the Doctrine of Chances", Biometrika 45, 293 – 315,.
[3] Bernoulli J. (1713): "Arts Conjectandi", Impensis Thurmisiorum Fratrum Basileae.
[4] Bernstein S. N. (1946): "The Theory of Probabilities (Russian)", Moscow, Leningrad.
[5] Dhritikesh Chakrabarty (2004): "A Theoretical Definition of Probability Based on Common Sense", Bulletin of Pure and Applied Sciences - E, 23E(2), 343 – 349. https://www.researchgate.net/publication/265315010_A_theoretical_definition_of_probability_based_on_common_sense .
[6] Dhritikesh Chakrabarty (2013): "A Journey for Understanding the Space of Research", National Seminar on Promotion of Research Culture in Enhancing Quality Higher Education, Held at Bimala Prasad Chaliha College in collaboration with Assam College Teachers' Association, June 26 – 28. DOI: 10.13140/RG.2.2.25678.23364 .
[7] Dhritikesh Chakrabarty (2020): "Definition / Formulation of Average from First Principle", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 9(2), 151 – 163. www.jecet.org . [DOI: 10.24214/jecet.C.9.2.15163]. .
[8] Dhritikesh Chakrabarty (2023): "Definition of Probability Based on Already Happened Outcomes: Application in Identifying Rainy and Non-Rainy Period", Partners Universal International Innovation Journal (PUIIJ), 01(04), 259 – 267. www.puiij.com . DOI:10.5281/zenodo.8282811.
[9] Dhritikesh Chakrabarty (2024): "Extension of Statistical Definition of Probability: Expected Number of Rainy Days in Indian Context", International Journal of Advanced Research in Science, Engineering and Technology, (ISSN: 2350 – 0328), 11(1), 21384 – 21393. www.ijarset.com .
[10] Jack B., Albert N. (1978): "A History of the Axiomatic Formulation of Probability from Borel to Kolmogorov", Archive for History of Exact Sciences, 18(2), 8.III. Springer. Stable URL: https://www.jstor.org/stable/i40049719 .
[11] Kolmogorov A. N. (1933): "Grunbegriffe der Wahrscheinlichkeits Rechnung", Ergeb. Math. And ihrer Grensg., 2, 62 – 88. (The Monograph Published by Springer, Berlin,1933).
[12] Koopman . B. O. (1940): "The Axioms and Algebra of Intuitive Probability", Ann. of Math.(2), 41, 269 – 292.
[13] Savage L. J. (1954): "The Foundations of Statistics", John Wiley, New York.
[14] Steele J. M. (2001): "International Encyclopedia of the Social & Behavioral Sciences".
[15] von Mises R. (1939): "Probability, Statistics and Truth", Mcmillan.
[16] von Mises R. (1941): "On the foundation of Probability and Statistics", Annals Mathematical Statistics, 12, 191 – 205.
[17] Wapner L. M, (2012): "Unexpected expectations: the curiosities of mathematical crystal ball", CRC Press, Taylor & Francis Group, London & New York.
[18] Wojciech Herer (1992): "Mathematical Expectation and Strong Law of Large Numbers for Random Variable with Values in a Metric Space of Negative Curvature, Probability and Mathematical Statistics, 13(1), 59 – 70.